Overview of the SYMPHONIE Model

SIROCCO Training Week

September 2026

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LEGOS

SYMPHONIE: an ocean model for currents, sea surface elevation, temperature, and salinity

General description

- Horizontal current
- Hydrostatic pressure
  - barotropic 2D, baroclinic 3D
- Incompressibility, continuity equation
- Vertical current
- Sea surface elevation
- Horizontal diffusion
  - numerical dispersion and diffusion
- Vertical diffusion
  - K-L scheme
  - K-eps scheme
- Bottom & surface conditions
- Sea water density
- Temperature, salinity

Overview of numerical methods

- Spatial derivative
  - C-grid
- Time stepping
- Sigma coordinate
- VQS coordinate
- External & internal modes

Equations for u and v

$$\underbrace{\frac{\partial u}{\partial t}}_{\text{local acceleration}} + \underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}}_{\text{advection}} - \underbrace{fv}_{\text{Coriolis}} = -\underbrace{\frac{1}{\rho_0}\frac{\partial p}{\partial x}}_{\text{pressure gradient}} + \underbrace{\mathcal{D}_u}_{\text{diffusion}}$$

$$\underbrace{\frac{\partial v}{\partial t}}_{\text{local acceleration}} + \underbrace{u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}}_{\text{advection}} + \underbrace{fu}_{\text{Coriolis}} = -\underbrace{\frac{1}{\rho_0}\frac{\partial p}{\partial y}}_{\text{pressure gradient}} + \underbrace{\mathcal{D}_v}_{\text{diffusion}}$$

Hydrostatic pressure

w equation ---> hydrostatic approximation ---> hydrostatic pressure

$$\underbrace{\cancel{\frac{\partial w}{\partial t}}}_{\text{local acceleration}} + \underbrace{\cancel{u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}}}_{\text{advection}} = -\underbrace{\frac{1}{\rho}\frac{\partial p}{\partial z}}_{\text{vertical pressure gradient}} - \underbrace{g}_{\text{gravity}} + \underbrace{\cancel{\mathcal{D}_w}}_{\text{diffusion}}$$

$$\underbrace{\frac{\partial p}{\partial z} = -\rho g}_{\text{hydrostatic balance}}$$

$$p(z) = \underbrace{p_{\text{atm}}}_{\substack{\text{atmospheric}\\\text{pressure}}} + \underbrace{g\int_z^{\eta} \rho(z')\,dz'}_{\substack{\text{hydrostatic}\\\text{pressure}}}$$

Separation into 2D barotropic and 3D baroclinic pressure

SYMPHONIE

$$\underbrace{\rho}_{\text{total density}} = \underbrace{\rho_0}_{\text{reference density}} + \underbrace{\rho'}_{\text{density anomaly}}$$

$$p(x,y,z) = p_{atm}(x,y,t) + \rho_0 g(\eta - z) + g\int_z^{\eta} \rho'(x,y,z',t)\,dz'$$

$$-\frac{1}{\rho_0}\frac{\partial p}{\partial x} = \underbrace{-\frac{1}{\rho_0}\frac{\partial p_{atm}}{\partial x}}_{\text{atmospheric pressure gradient}} - \underbrace{g\frac{\partial \eta}{\partial x}}_{\text{barotropic pressure gradient}} - \underbrace{\frac{g}{\rho_0}\frac{\partial}{\partial x}\left(\int_z^{\eta}\rho'\,dz'\right)}_{\text{baroclinic pressure gradient}}$$

$$\frac{Du}{Dt} - fv = \underbrace{-\frac{1}{\rho_0}\frac{\partial p_{atm}}{\partial x}}_{\text{atmospheric pressure}} - \underbrace{g\frac{\partial \eta}{\partial x}}_{\text{2D / barotropic}} - \underbrace{\frac{g}{\rho_0}\frac{\partial}{\partial x}\left(\int_z^{\eta}\rho'\,dz'\right)}_{\text{3D / baroclinic}} + \mathcal{D}_u$$

Obtain w from continuity equation

Incompressibility assumption

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0}$$

$$\boxed{\frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}}$$

$$\boxed{w(z) = w(-h) - \int_{-h}^{z} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dz'}$$

Horizontal velocity divergence $\longrightarrow$ vertical velocity

Bottom and surface kinematic conditions:

$$\boxed{w(-h) = -u_b \frac{\partial h}{\partial x} - v_b \frac{\partial h}{\partial y}}$$

$$\boxed{w_\eta = \frac{\partial \eta}{\partial t} + u_\eta \frac{\partial \eta}{\partial x} + v_\eta \frac{\partial \eta}{\partial y}}$$

Obtain sea surface elevation from continuity equation

Incompressibility assumption

$$\boxed{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0}$$

- Vertical integral from the bottom to the surface
- Leibniz's rule
- Bottom and surface kinematic conditions
- Add air-sea exchanges, river inputs

$$\boxed{\frac{\partial \eta}{\partial t}+\frac{\partial}{\partial x}\left(\int_{-h}^{\eta} u\,dz\right)+\frac{\partial}{\partial y}\left(\int_{-h}^{\eta} v\,dz\right)=\underbrace{P-E+R}_{\text{freshwater fluxes}}}$$

Diffusion

- Horizontal diffusion is primarily a numerical issue
  - Numerical dispersion of advection
- Vertical diffusion is more of a physical issue
  - Vertical turbulence (wind, bottom friction…)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \boxed{\mathcal{D}_u}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \boxed{\mathcal{D}_v}$$

$$\mathcal{D}_u = \underbrace{\frac{\partial}{\partial x}\left(A_H\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_H\frac{\partial u}{\partial y}\right)}_{\text{horizontal diffusion}} + \underbrace{\frac{\partial}{\partial z}\left(K_m\frac{\partial u}{\partial z}\right)}_{\text{vertical diffusion}}$$

$$\mathcal{D}_v = \underbrace{\frac{\partial}{\partial x}\left(A_H\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_H\frac{\partial v}{\partial y}\right)}_{\text{horizontal diffusion}} + \underbrace{\frac{\partial}{\partial z}\left(K_m\frac{\partial v}{\partial z}\right)}_{\text{vertical diffusion}}$$

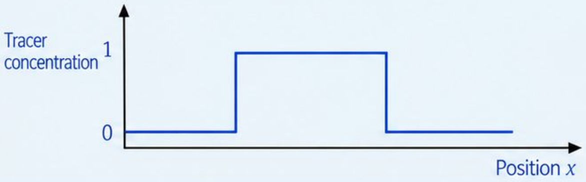
Numerical dispersion in advection schemes

Example with a step (square pulse)

SYMPHONIE

1 Initial condition

A step (square pulse) containing many spatial scales.



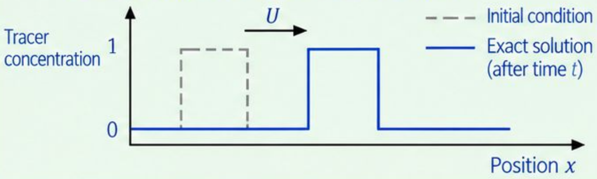
Pure advection

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} = 0$$

Transport at a constant velocity U

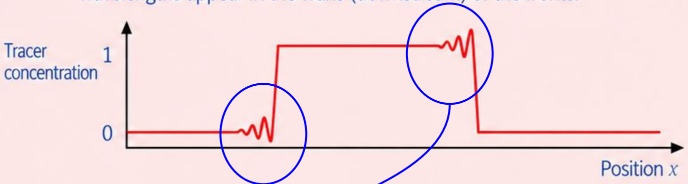
2 Exact solution

The step is translated without change of shape (all wavelengths travel at the same velocity U).



3 Numerical solution with dispersion

The step is transported at the correct mean speed, but small short wavelengths appear in the wake (downstream) of the fronts.



4 Numerical phase speed

The numerical phase speed is lower than the exact value and approaches it from below as the wavelength increases.

$$\mathcal{D}_u = \underbrace{\frac{\partial}{\partial x}\left(\boxed{A_H}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\boxed{A_H}\frac{\partial u}{\partial y}\right)}_{\text{horizontal diffusion}} + \underbrace{\frac{\partial}{\partial z}\left(K_m\frac{\partial u}{\partial z}\right)}_{\text{vertical diffusion}}$$

$$\mathcal{D}_v = \underbrace{\frac{\partial}{\partial x}\left(\boxed{A_H}\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\boxed{A_H}\frac{\partial v}{\partial y}\right)}_{\text{horizontal diffusion}} + \underbrace{\frac{\partial}{\partial z}\left(K_m\frac{\partial v}{\partial z}\right)}_{\text{vertical diffusion}}$$

Numerical dispersion does not remove energy, but it modifies the phase speed, especially for short wavelengths, which generates small dispersive oscillations in the wake of sharp gradients.
Advection schemes are designed to minimize numerical dispersion.

$$\boxed{A_H = \alpha|\mathbf{U}|\Delta x} \qquad \alpha \sim O(0.1-1)$$

The issue of horizontal diffusion is to eliminate the numerical noise caused by errors of numerical advection dispersion by selecting an appropriate diffusion coefficient value.

Vertical diffusion: 2 turbulence models available

K-L Turbulence Model

$$K_m = C_m \, l \sqrt{k}$$

$$\frac{\partial k}{\partial t} + u\frac{\partial k}{\partial x} + v\frac{\partial k}{\partial y} + w\frac{\partial k}{\partial z} = \underbrace{P_s}_{\text{shear production}} + \underbrace{P_b}_{\text{buoyancy production}} - \underbrace{\varepsilon}_{\text{dissipation}} + \underbrace{\frac{\partial}{\partial z}\left(K_k \frac{\partial k}{\partial z}\right)}_{\text{vertical turbulent diffusion}}$$

$$l = \min(l_u, l_d)$$

$$\varepsilon = C_\varepsilon \frac{k^{3/2}}{l}$$

$$\int_z^{z+l_u} \frac{g}{\rho_0}\left[\rho(z) - \rho(z')\right] dz' = k(z)$$

$$\int_{z-l_d}^{z} \frac{g}{\rho_0}\left[\rho(z') - \rho(z)\right] dz' = k(z)$$

- The mixing coefficient depends on *k*, the turbulent kinetic energy, and *l* the mixing length
- The dissipation rate depends (simply) on *k* and *l*
- The mixing length is limited by buoyancy (Gaspar et al., 1990)

Vertical diffusion: 2 turbulence models available

SYMPHONIE

K-eps Turbulence Model

$$\boxed{K_m = S_m \, l \sqrt{k}}$$

$$\boxed{l = C_\varepsilon \frac{k^{3/2}}{\varepsilon}}$$

$$\frac{\partial k}{\partial t} + u\frac{\partial k}{\partial x} + v\frac{\partial k}{\partial y} + w\frac{\partial k}{\partial z} = \underbrace{P_s}_{\text{shear production}} + \underbrace{P_b}_{\text{buoyancy production}} - \underbrace{\varepsilon}_{\text{dissipation}} + \underbrace{\frac{\partial}{\partial z}\left(K_k \frac{\partial k}{\partial z}\right)}_{\text{vertical turbulent diffusion}}$$

$$\frac{\partial \varepsilon}{\partial t} + u\frac{\partial \varepsilon}{\partial x} + v\frac{\partial \varepsilon}{\partial y} + w\frac{\partial \varepsilon}{\partial z} = C_{\varepsilon 1}\frac{\varepsilon}{k}(P_s + P_b) - C_{\varepsilon 2}\frac{\varepsilon^2}{k} + \frac{\partial}{\partial z}\left(K_\varepsilon \frac{\partial \varepsilon}{\partial z}\right)$$

- The formulation of *Km* is improved with *Sm*, a stability function
- The dissipation rate is predicted by an equation of higher complexity than the K-L scheme

Surface boundary condition

$$K_m \left.\frac{\partial u}{\partial z}\right|_{z=\eta} = \frac{\tau_{wx}}{\rho_0}$$

$$K_m \left.\frac{\partial v}{\partial z}\right|_{z=\eta} = \frac{\tau_{wy}}{\rho_0}$$

$$\boldsymbol{\tau}_w = \rho_a C_D^{air} |\mathbf{U}_{10} - \mathbf{u}_s| (\mathbf{U}_{10} - \mathbf{u}_s)$$

$$\mathcal{D}_u = \underbrace{\frac{\partial}{\partial x}\left(A_H \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_H \frac{\partial u}{\partial y}\right)}_{\text{horizontal diffusion}} + \underbrace{\frac{\partial}{\partial z}\left(K_m \frac{\partial u}{\partial z}\right)}_{\text{vertical diffusion}}$$

$$\mathcal{D}_v = \underbrace{\frac{\partial}{\partial x}\left(A_H \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_H \frac{\partial v}{\partial y}\right)}_{\text{horizontal diffusion}} + \underbrace{\frac{\partial}{\partial z}\left(K_m \frac{\partial v}{\partial z}\right)}_{\text{vertical diffusion}}$$

- Wind stress is introduced through a boundary condition on the vertical turbulent momentum flux
- Wind stress is calculated using bulk formulas

Bottom boundary condition

$$K_m \frac{\partial u}{\partial z}\bigg|_{z=-h} = C_D |\mathbf{u}_b| u_b$$

$$K_m \frac{\partial v}{\partial z}\bigg|_{z=-h} = C_D |\mathbf{u}_b| v_b$$

$$\mathcal{D}_u = \underbrace{\frac{\partial}{\partial x}\left(A_H \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_H \frac{\partial u}{\partial y}\right)}_{\text{horizontal diffusion}} + \underbrace{\frac{\partial}{\partial z}\left(\boxed{K_m \frac{\partial u}{\partial z}}\right)}_{\text{vertical diffusion}}$$

$$\mathcal{D}_v = \underbrace{\frac{\partial}{\partial x}\left(A_H \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_H \frac{\partial v}{\partial y}\right)}_{\text{horizontal diffusion}} + \underbrace{\frac{\partial}{\partial z}\left(\boxed{K_m \frac{\partial v}{\partial z}}\right)}_{\text{vertical diffusion}}$$

$$C_D = \frac{\kappa^2}{\left[\ln(d_1/z_0)\right]^2} \qquad \underbrace{\kappa \simeq 0.4}_{\text{von Kármán constant}}$$

- Friction is introduced through a boundary condition on the vertical turbulent momentum flux
- The drag coefficient *CD* is derived from the assumption of a logarithmic current profile and a roughness length *z0*
- The bottom stress is calculated at a distance *d1* from the bottom (first vertical grid level providing *ub* and *vb*)

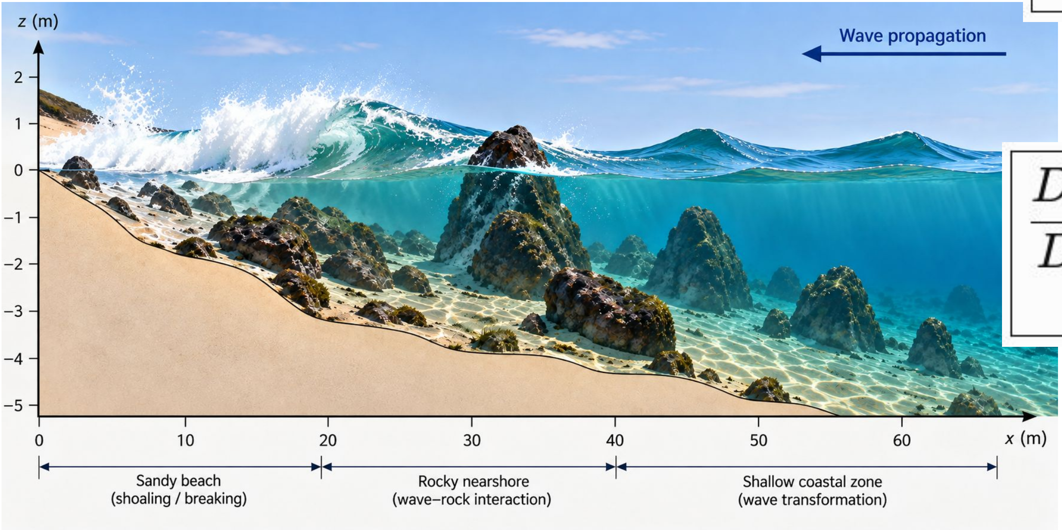
3D effect of macro-roughness (rocks, submerged structures, vegetation, etc.)

- 3D quadratic drag coefficient (friction)
- Inertial force (total volume partially occupied by a solid volume)
- Currently available only in the non-hydrostatic version for phase-resolved wave modeling

$$\mathbf{F}_{can} = \underbrace{\frac{1}{2} C_D\, a\, |\mathbf{u}|\, \mathbf{u}}_{\text{quadratic drag}} + \underbrace{C_A\, \phi\, \frac{\partial \mathbf{u}}{\partial t}}_{\text{inertial force}}$$

$$\mathbf{F}_{drag} = \frac{1}{2} C_D\, a\, |\mathbf{u}|\, \mathbf{u}$$

$$\mathbf{F}_{inertia} = C_A\, \phi\, \frac{\partial \mathbf{u}}{\partial t}$$



$$\frac{Du}{Dt} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \mathcal{D}_u - \underbrace{F_{can,x}}_{\text{canopy force}}$$

Hydrostatic pressure and water density

$$p(z) = \underbrace{p_{\text{atm}}}_{\substack{\text{atmospheric}\\ \text{pressure}}} + \underbrace{g\int_z^{\eta} \rho(z')\,dz'}_{\substack{\text{hydrostatic}\\ \text{pressure}}}$$

- The baroclinic pressure gradient depends on seawater density, which is computed from temperature, salinity and pressure using the equation of state

$$\boxed{\rho = \rho(T, S, p)}$$

- To first order, seawater density decreases with increasing temperature and increases with increasing salinity

$$\boxed{\alpha = -\frac{1}{\rho}\frac{\partial \rho}{\partial T}, \qquad \beta = \frac{1}{\rho}\frac{\partial \rho}{\partial S}}$$

$$\boxed{T\uparrow \Rightarrow \rho\downarrow \qquad S\uparrow \Rightarrow \rho\uparrow \qquad p\uparrow \Rightarrow \rho\uparrow}$$

- Beyond this first-order dependence, the TEOS-10 equation of state involves a high-order polynomial representation of the thermodynamic properties of seawater

- Determining density requires calculating temperature and salinity

Equations for T and S

- Transport equations: advection and diffusion
- Solar forcing term in the case of T

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \underbrace{\nabla_H \cdot (K_H \nabla_H T)}_{\text{horizontal diffusion}} + \underbrace{\frac{\partial}{\partial z}\left(K_\rho \frac{\partial T}{\partial z}\right)}_{\text{vertical turbulent diffusion}} + \underbrace{\frac{1}{\rho_0 C_p}\frac{\partial Q_{sw}}{\partial z}}_{\text{solar heating}}$$

$$\frac{\partial S}{\partial t} + u\frac{\partial S}{\partial x} + v\frac{\partial S}{\partial y} + w\frac{\partial S}{\partial z} = \underbrace{\nabla_H \cdot (K_H \nabla_H S)}_{\text{horizontal diffusion}} + \underbrace{\frac{\partial}{\partial z}\left(K_S \frac{\partial S}{\partial z}\right)}_{\text{vertical turbulent diffusion}}$$

Surface boundary conditions for T and S

- Air/Sea exchanges are introduced through a boundary condition on the vertical turbulent heat flux in temperature equation

$$\boxed{K_\rho \frac{\partial T}{\partial z}\bigg|_{z=\eta} = \frac{Q_{net}}{\rho_0 C_p}}$$

$$\boxed{Q_{net} = \underbrace{Q_{LW}}_{\text{longwave radiation}} + \underbrace{Q_{latent}}_{\text{latent heat flux}} + \underbrace{Q_{sensible}}_{\text{sensible heat flux}}}$$

- No salt passes through the surface of the ocean

- Changes in surface salinity result indirectly from the inputs (vs. outputs) of freshwater at the surface, which decrease (vs. increase) salinity without changing the total salt content

$$\boxed{\frac{\partial \eta}{\partial t} + \nabla_H \cdot \left(\int_{-h}^{\eta} \mathbf{u}_H \, dz \right) = \underbrace{P - E + R}_{\text{freshwater flux}}}$$

$$\boxed{\text{Freshwater flux} \longrightarrow \text{water volume change} \longrightarrow \text{surface salinity change}}$$

An Overview of Numerical Methods

- Spatial derivative
  - C-grid
- Time stepping
- Sigma coordinate
- VQS coordinate

$$\phi_{i\pm1/2} = \phi_i \pm \frac{\Delta x}{2}\phi_i' + \frac{\Delta x^2}{8}\phi_i'' \pm \frac{\Delta x^3}{48}\phi_i''' + \cdots$$

$$\frac{\phi_{i+1/2} - \phi_{i-1/2}}{\Delta x} = \phi_i' + \boxed{\frac{\Delta x^2}{24}\phi_i'''} + O(\Delta x^4)$$

leading error term

$$\boxed{\left.\frac{\partial \phi}{\partial x}\right|_i \simeq \frac{\phi_{i+1/2} - \phi_{i-1/2}}{\Delta x}}$$

Spatial derivatives are calculated using second-order centered finite-difference schemes

The continuity equation case

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$$\left.\frac{\partial u}{\partial x}\right|_{i,j,k} = \frac{u_{i+\frac{1}{2},j,k} - u_{i-\frac{1}{2},j,k}}{\Delta x} + \underbrace{O(\Delta x^2)}_{\text{truncation error}}$$

$$\left.\frac{\partial v}{\partial y}\right|_{i,j,k} = \frac{v_{i,j+\frac{1}{2},k} - v_{i,j-\frac{1}{2},k}}{\Delta y} + \underbrace{O(\Delta y^2)}_{\text{truncation error}}$$

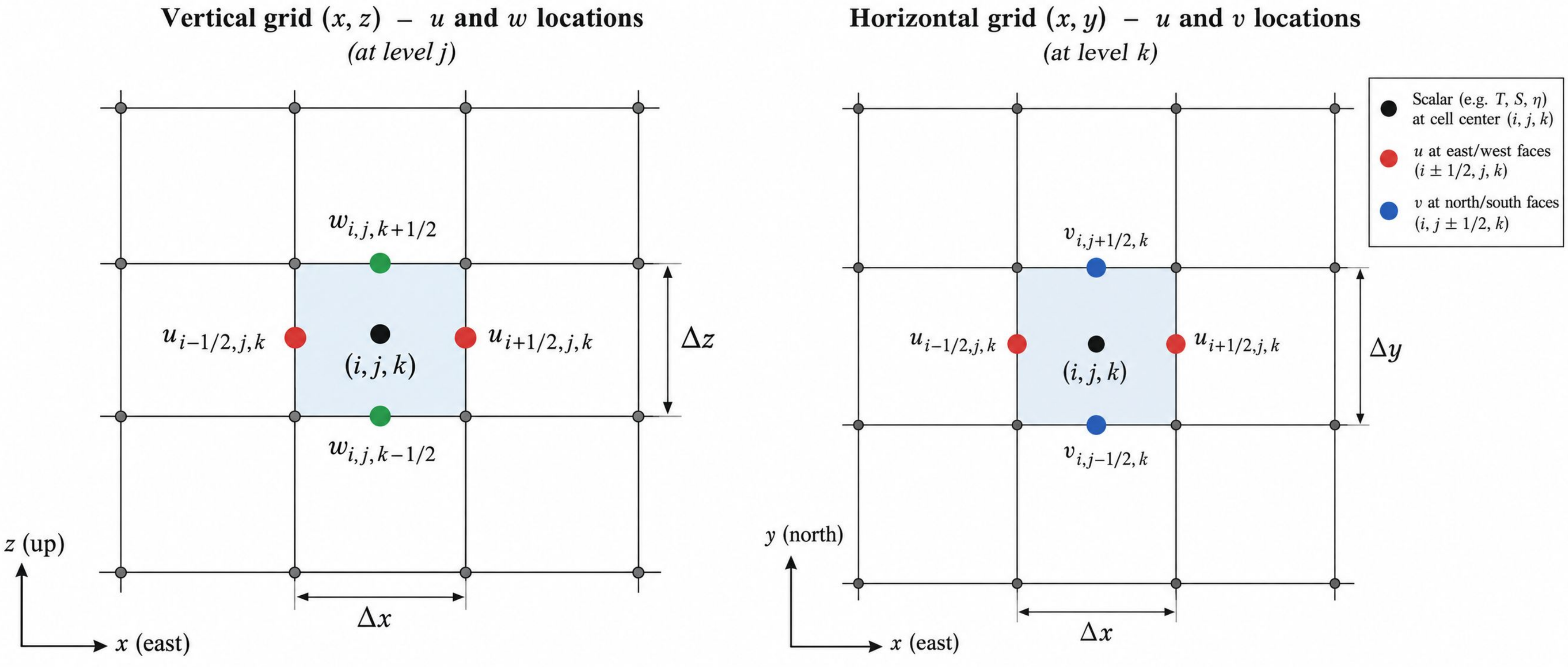
$$\left.\frac{\partial w}{\partial z}\right|_{i,j,k} = \frac{w_{i,j,k+\frac{1}{2}} - w_{i,j,k-\frac{1}{2}}}{\Delta z} + \underbrace{O(\Delta z^2)}_{\text{truncation error}}$$

The simplicity of this expression is made possible by the clever spatial distribution of the C grid

$$\frac{u_{i+\frac{1}{2},j,k} - u_{i-\frac{1}{2},j,k}}{\Delta x} + \frac{v_{i,j+\frac{1}{2},k} - v_{i,j-\frac{1}{2},k}}{\Delta y} + \frac{w_{i,j,k+\frac{1}{2}} - w_{i,j,k-\frac{1}{2}}}{\Delta z} = 0$$

C grid

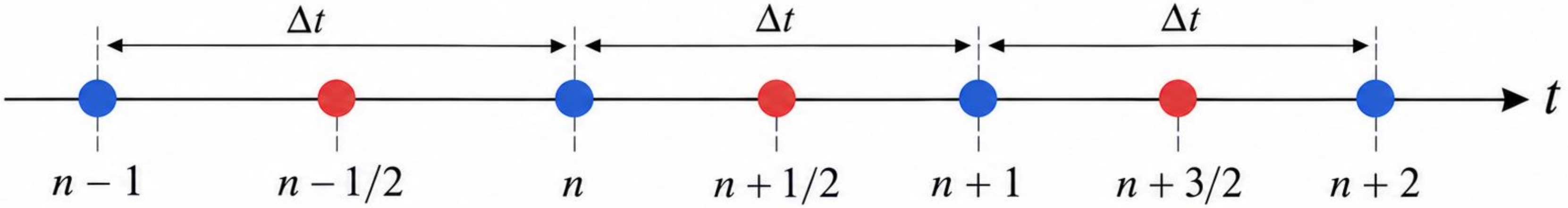
- The components *u, v,* and *w* are positioned at the edges of the computational cells
- Since divergence is the most frequently used operation in the model (continuity equation, advection, diffusion, etc.), the C grid is optimal for most of the terms calculated by the model



Hybrid Leap-Frog / Forward–Backward time stepping (time staggering)

- Velocities (u, v, w) : integer times
- Tracers $(T, S, \ldots)$: half times

This hybrid scheme combines the advantages of Leap-Frog (Coriolis & advection centering => stability) with those of the forward-backward scheme (greater gravitational wave stability), resulting in a larger time step



Velocities (u, v, w) : integer times

$$\left.\frac{\partial u}{\partial t}\right|^{n} \approx \frac{u^{n+1} - u^{n-1}}{2\Delta t}$$

(second-order accurate in time)

Tracers $(T, S, \ldots)$: half times

$$\left.\frac{\partial \phi}{\partial t}\right|^{n+1/2} \approx \frac{\phi^{n+3/2} - \phi^{n-1/2}}{2\Delta t}$$

(second-order accurate in time)

Sigma vertical coordinate

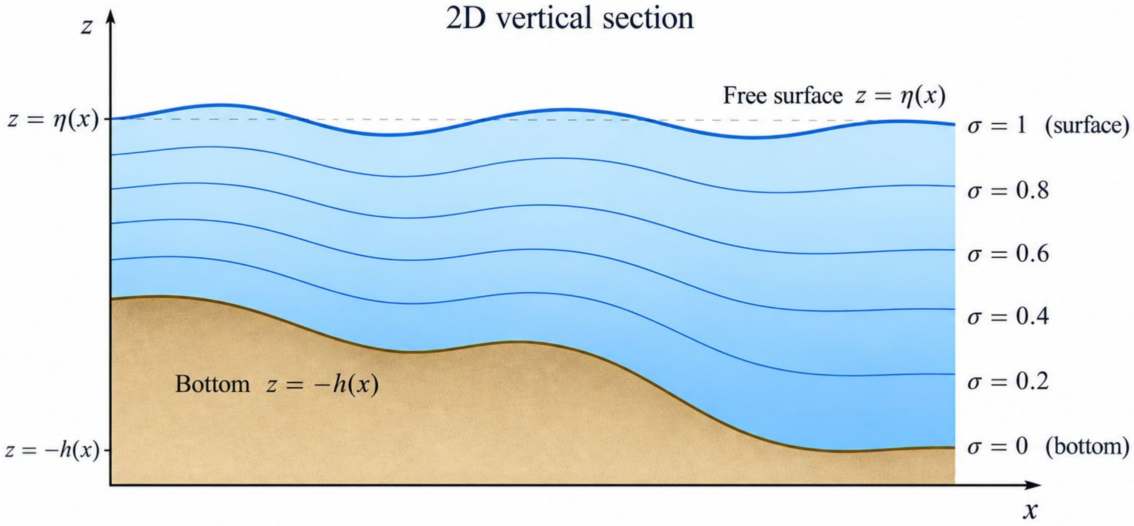
The sigma coordinate follows the bottom and the surface, and maintains a constant number of vertical levels

Advantages

- Accurately follows the bathymetry: there are no steps on the seafloor.
- The same number of vertical levels regardless of depth.
- Good relative resolution near the seafloor at shallow depths, which is particularly useful for bottom boundary layers.
- Natural representation of seafloor currents, friction, coastal processes, and interactions with topography

Applications: particularly well suited for coastal and regional oceanography: continental shelves, coastal zones, estuaries, lagoons, tides, coastal circulation, current–topography interactions, and boundary layer process

Sigma (σ) vertical coordinate (SYMPHONIE convention)



$$\sigma = \frac{h(x) + z}{h(x) + \eta(x)} \qquad 0 \le \sigma \le 1$$

$\sigma = 0$ at the bottom $(z = -h)$

$\sigma = 1$ at the free surface $(z = \eta)$

$\sigma =$ constant follows the topography and the surface

Spatial derivative with sigma coordinate

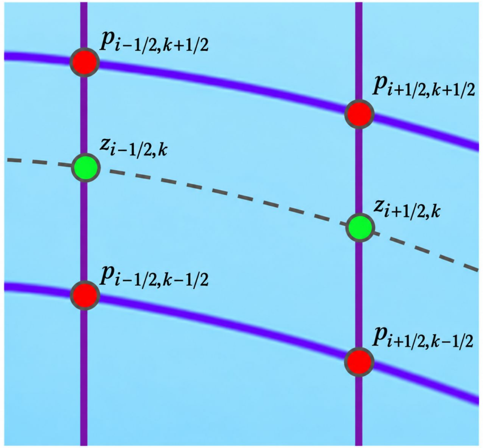
Main drawback

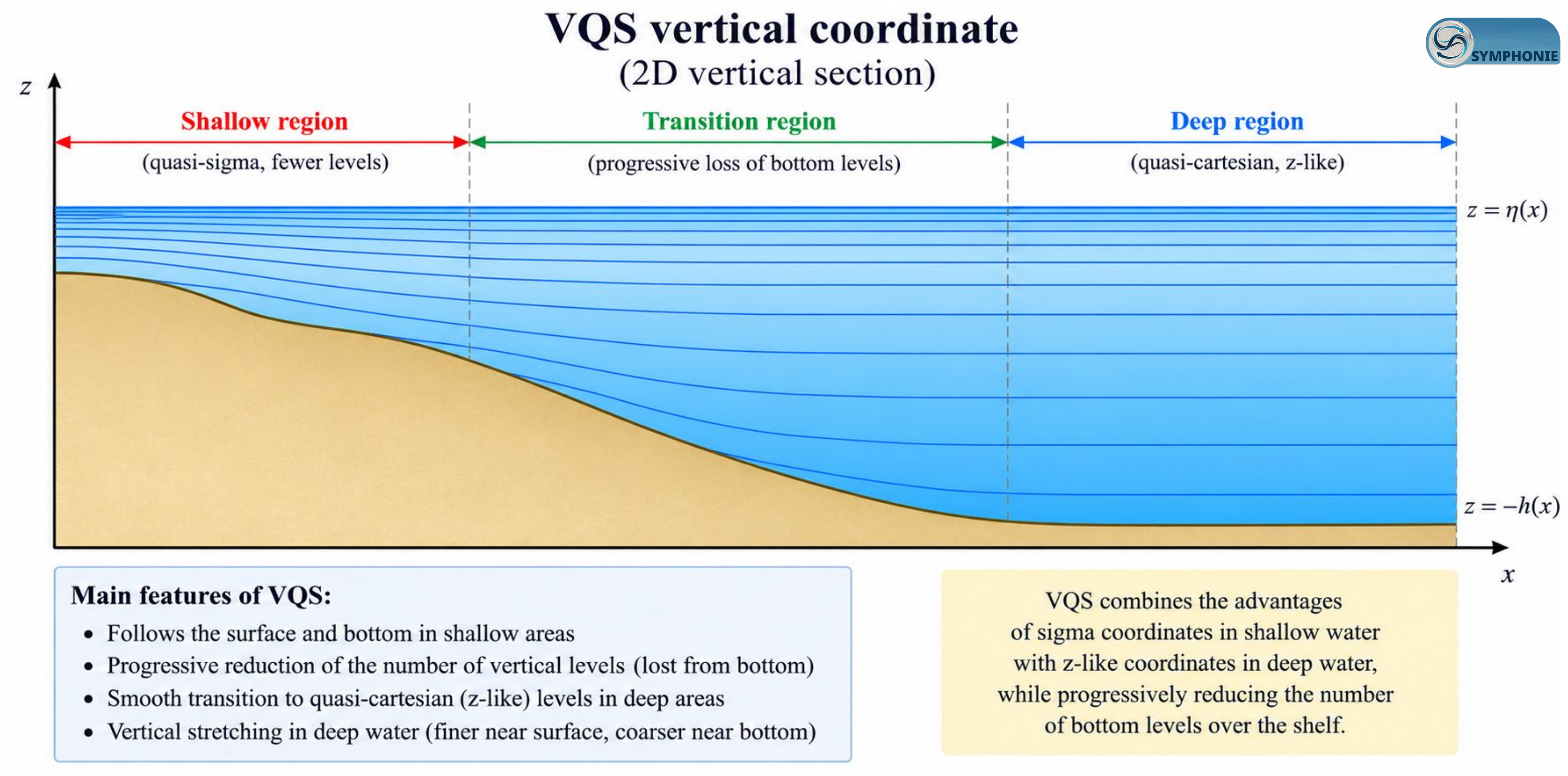
- Sloping sigma levels complicate the computation of horizontal pressure gradients
- This can generate significant pressure-gradient truncation errors, especially over steep topography and strong stratification
- Similar geometrical issues also affect other key operators, notably advection and diffusion

$$\boxed{\frac{\partial p}{\partial x} = \underbrace{\frac{\partial p}{\partial x^*}}_{\substack{\text{gradient along}\\ \text{a sigma level}}} - \underbrace{\frac{\partial z}{\partial x^*}\frac{\partial p}{\partial z}}_{\substack{\text{vertical-coordinate}\\ \text{correction}}}}$$

$$\simeq \frac{\frac{1}{2}\left(p_{i+\frac{1}{2},k+\frac{1}{2}} + p_{i+\frac{1}{2},k-\frac{1}{2}}\right) - \frac{1}{2}\left(p_{i-\frac{1}{2},k+\frac{1}{2}} + p_{i-\frac{1}{2},k-\frac{1}{2}}\right)}{\Delta x}$$

$$- \frac{z_{i+\frac{1}{2},k} - z_{i-\frac{1}{2},k}}{\Delta x} \frac{\frac{1}{2}\left(p_{i-\frac{1}{2},k+\frac{1}{2}} + p_{i+\frac{1}{2},k+\frac{1}{2}}\right) - \frac{1}{2}\left(p_{i-\frac{1}{2},k-\frac{1}{2}} + p_{i+\frac{1}{2},k-\frac{1}{2}}\right)}{\Delta z}$$





- Reduces truncation errors in the pressure gradient
- Reduces vertical numerical erosion in diffusion operations

Time splitting, external & internal modes

$$u(x,y,z,t) = \underbrace{\overline{u}(x,y,t)}_{\text{barotropic mode}} + \underbrace{u'(x,y,z,t)}_{\text{baroclinic mode}}$$

$$v(x,y,z,t) = \underbrace{\overline{v}(x,y,t)}_{\text{barotropic mode}} + \underbrace{v'(x,y,z,t)}_{\text{baroclinic mode}}$$

$$\underbrace{\overline{u}}_{\substack{\text{depth-averaged}\\ \text{velocity}}} = \frac{1}{H}\int_{-h}^{\eta} u\, dz$$

$$\underbrace{\overline{v}}_{\substack{\text{depth-averaged}\\ \text{velocity}}} = \frac{1}{H}\int_{-h}^{\eta} v\, dz$$

$$H = h + \eta$$

- Separates the fast depth-averaged (barotropic) dynamics from the slower 3D baroclinic dynamics
- This allows the two modes to be integrated with different time steps (time splitting), greatly reducing computational cost

Equations for barotropic transport U and V

$$\underbrace{\frac{\partial U}{\partial t}}_{\substack{\text{barotropic}\\\text{acceleration}}} + \underbrace{\frac{\partial}{\partial x}\left(\frac{U^2}{H}\right) + \frac{\partial}{\partial y}\left(\frac{UV}{H}\right)}_{\text{advection}} - \underbrace{fV}_{\text{Coriolis}} = - \underbrace{gH\frac{\partial \eta}{\partial x}}_{\substack{\text{barotropic}\\\text{pressure gradient}}} + \underbrace{F_U}_{\substack{\text{3D residual terms, stresses,}\\\text{diffusion, other forcing}}}$$

$$\underbrace{U}_{\substack{\text{barotropic}\\\text{transport}}} = \int_{-h}^{\eta} u\,dz = H\,\overline{u}$$

$$\underbrace{\frac{\partial V}{\partial t}}_{\substack{\text{barotropic}\\\text{acceleration}}} + \underbrace{\frac{\partial}{\partial x}\left(\frac{UV}{H}\right) + \frac{\partial}{\partial y}\left(\frac{V^2}{H}\right)}_{\text{advection}} + \underbrace{fU}_{\text{Coriolis}} = - \underbrace{gH\frac{\partial \eta}{\partial y}}_{\substack{\text{barotropic}\\\text{pressure gradient}}} + \underbrace{F_V}_{\substack{\text{3D residual terms, stresses,}\\\text{diffusion, other forcing}}}$$

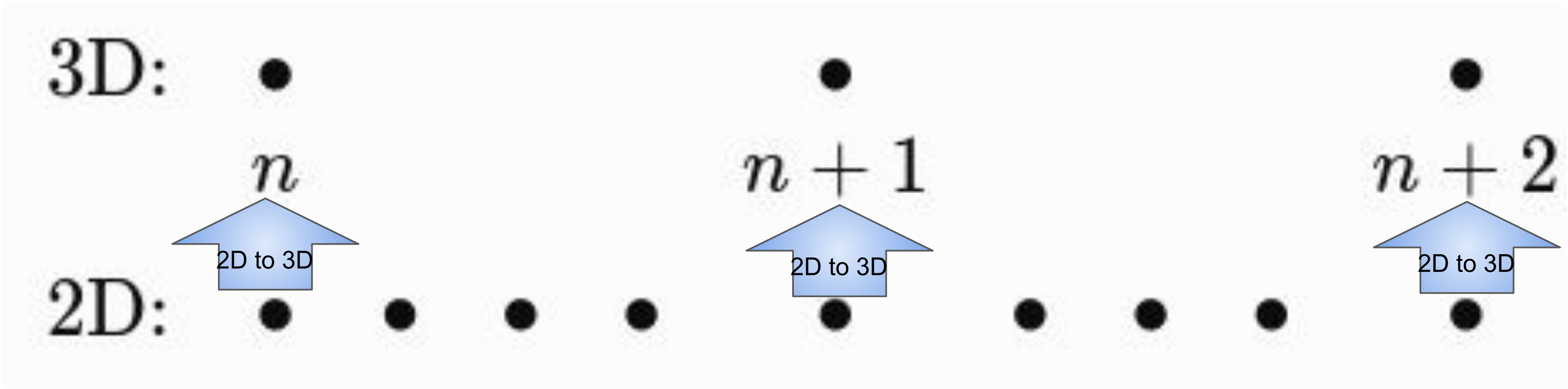
$$\underbrace{V}_{\substack{\text{barotropic}\\\text{transport}}} = \int_{-h}^{\eta} v\,dz = H\,\overline{v}$$

- These 2D equations are solved using a small time step
- The residual terms contain the nonlinear advective terms of the baroclinic current and the 3D pressure gradient dependent on the density anomaly (among other terms)

$$\underbrace{-\left[\frac{\partial}{\partial x}\left(\int_{-h}^{\eta} u'^2\,dz\right) + \frac{\partial}{\partial y}\left(\int_{-h}^{\eta} u'v'\,dz\right)\right]}_{\substack{\text{3D baroclinic}\\\text{nonlinear contribution}}}$$

$$\underbrace{\frac{g}{\rho_0}\int_{-h}^{\eta}\frac{\partial}{\partial x}\left(\int_{z''}^{\eta}\rho'(z')\,dz'\right)dz''}_{\substack{\text{baroclinic pressure}\\\text{gradient contribution}}}$$

Internal and External Mode Coupling



- The internal mode is actually a "classical" equation (without a distinction between modes) whose vertical average is corrected by the vertically averaged current calculated by the external mode:

$$\underbrace{u^{n+1}(z)}_{\substack{\text{corrected}\\ \text{3D current}}} = \underbrace{u^{*}(z)}_{\substack{\text{provisional}\\ \text{3D current}}} - \underbrace{\overline{u^{*}}}_{\substack{\text{provisional}\\ \text{depth-mean current}}} + \underbrace{\bar{u}^{2D}}_{\substack{\text{2D external-mode}\\ \text{current}}}$$

Thank you for your attention