Altimétrie spatiale

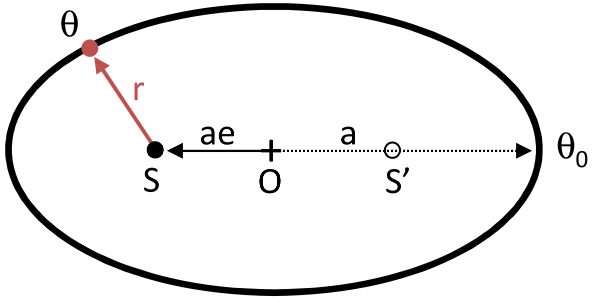
Mécanique Spatiale

Richard Biancale / CNES

The Kepler's laws

Johannes Kepler
1571-1630

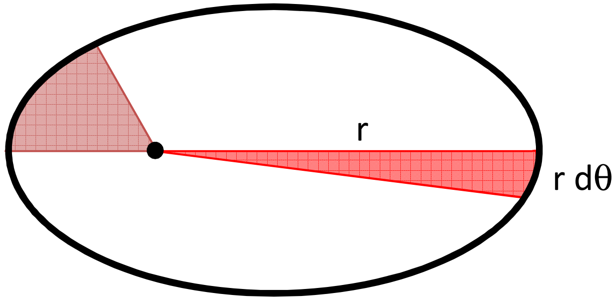
First law (1609) : The orbit of each planet is an ellipse whose one focus is the Sun.



$$r = \frac{\left\|\overline{C}\right\|^2}{GM}\frac{1}{1+e\cos(\theta-\theta_0)}$$

$$\overline{C} = \overline{r}\times\dot{\overline{r}} \quad \textit{kinetic} \quad \textit{momentum}$$

Second law (1609) : The line joining a planet and the Sun sweeps out equal areas during equal time intervals.

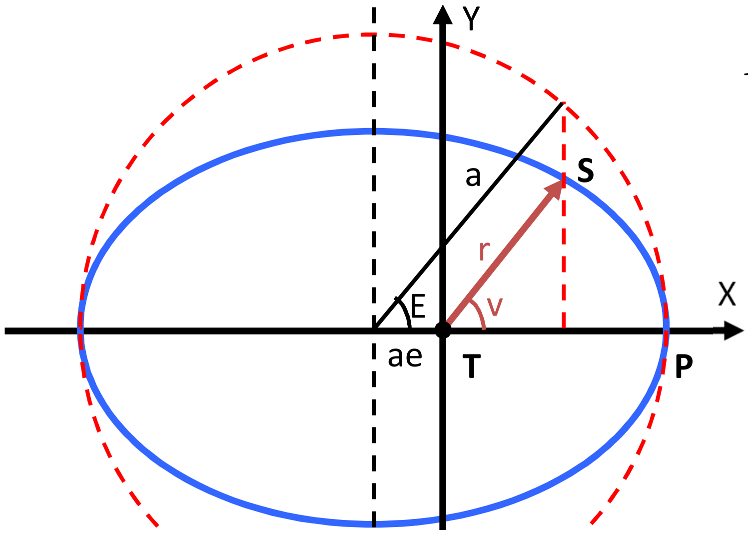


$$dA = \frac{1}{2}r^2 d\theta \quad \Rightarrow \quad \frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{1}{2}\left\|\overline{C}\right\|$$

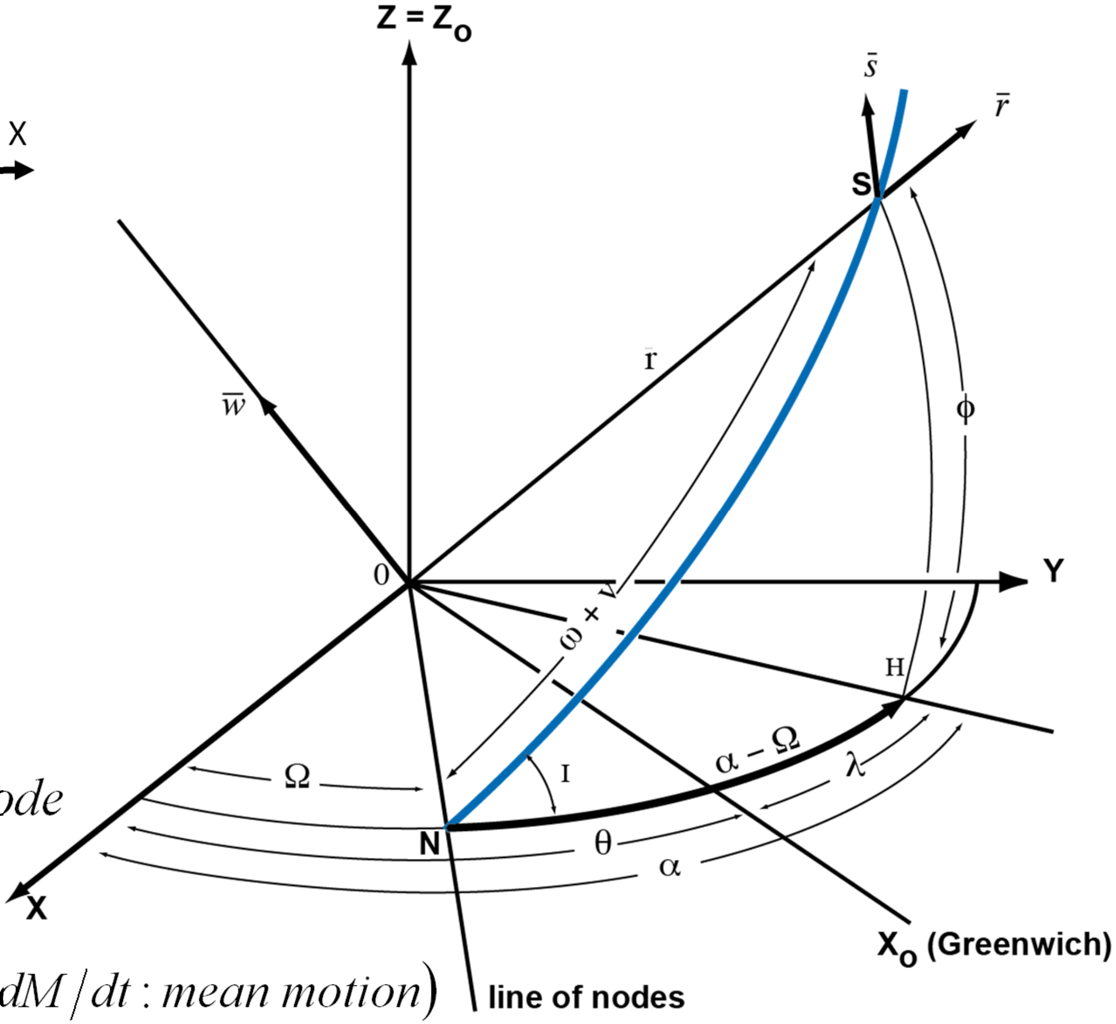
Third law (1619) : The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit :

$$n^2 a^3 = GM \quad \textit{with} \quad n = 2\pi/T \quad \Leftrightarrow \quad T = 2\pi\sqrt{\frac{a^3}{GM}}$$

The Keplerian elements



Kepler's equation : $M = E - e \sin E$



Keplerian orbital elements

a: semi-major axis

e: eccentricity $\left(b = a\sqrt{1-e^2}\right)$

i: inclination

Ω: argument of the ascending node

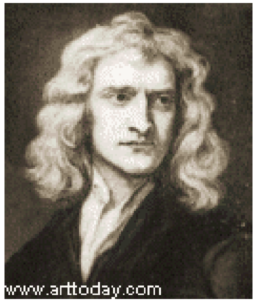
ω: argument of perigee

$M=n(t-t_0)$*: mean anomaly* $\left(n = dM/dt : \text{mean motion}\right)$

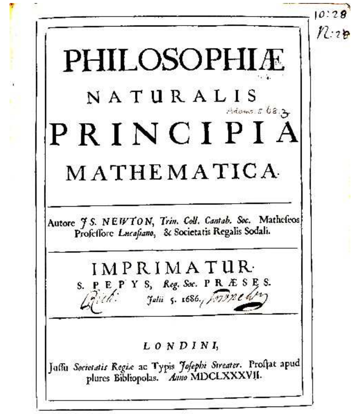
The Newton's laws (1687)

The Kepler's laws are descriptive. Their origin results from the law of universal gravitation as stated by Isaac Newton in 1687 in his « Principia Mathematica »:

$$\boxed{\ddot{\bar{r}} = -GM\,\frac{\bar{r}}{r^3}}$$



Isaac Newton
1643-1727


PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA.

Autore J S. NEWTON, Trin. Coll. Cantab. Soc. Matheseos Professore Lucasiano, & Societatis Regalis Sodali.

IMPRIMATUR.
S. PEPYS, Reg. Soc. PRÆSES.
Julii 5. 1686.

LONDINI,
Jussu Societatis Regiæ ac Typis Josephi Streater. Prostat apud plures Bibliopolas. Anno MDCLXXXVII.

First law: if an object experiences no net force, then its velocity is constant; the object is either at rest (if its velocity is zero), or it moves in a straight line with constant speed (if its velocity is non zero).

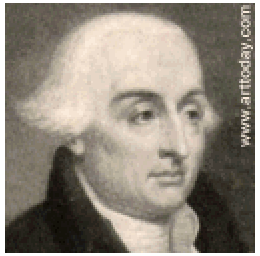
Second law: the acceleration **A** of a body is parallel and directly proportional to the net force **F** acting on the body, is in the direction of the net force, and is inversely proportional to the mass *m* of the body, i.e. $\mathbf{F} = m\mathbf{A}$.

Third law: when a first body exerts a force $\mathbf{F}_1$ on a second body, the second body simultaneously exerts a force $\mathbf{F}_2 = -\mathbf{F}_1$ on the first body. This means that $\mathbf{F}_1$ and $\mathbf{F}_2$ are equal in magnitude and opposite in direction

The perturbed motion *(described by Joseph-Louis Lagrange in his treatise on “**Mécanique Analytique**”, 1788)*

The equation of the elliptical motion is solution of the 2-body problem:

$$\ddot{\bar{r}} = -GM\,\frac{\bar{r}}{r^3}$$

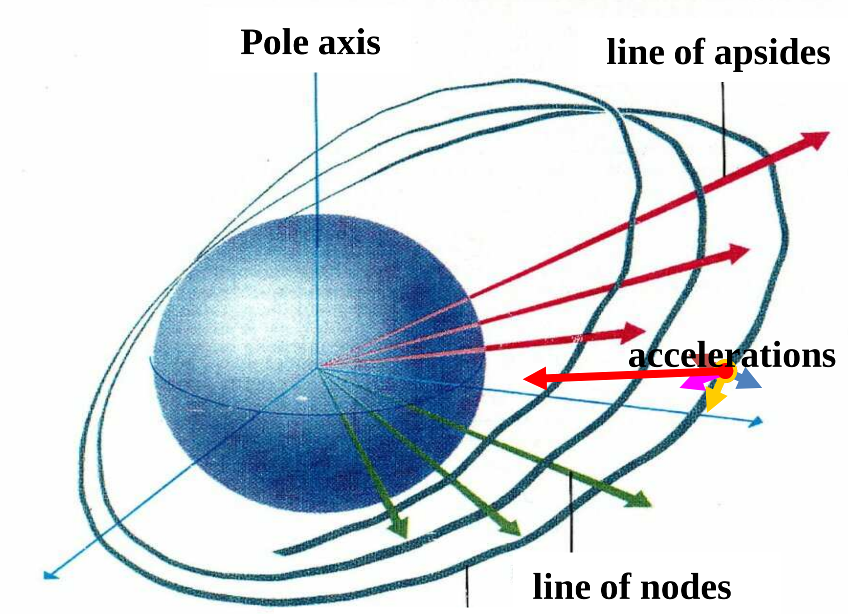


whose constant Keplerian elements *a*, *e*, *i*, *Ω*, *ω* and *n=dM/dt* are solution.

Adding complementary accelerations $\bar{\gamma}$ generates perturbations of the elliptic orbit:

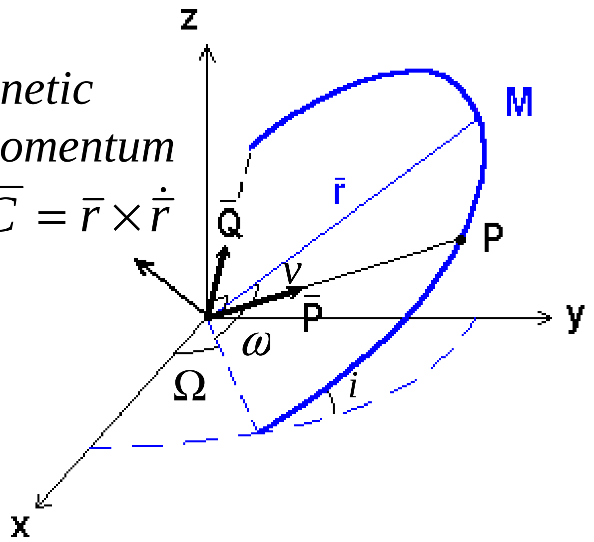
$$\ddot{\bar{r}} = -GM\,\frac{\bar{r}}{r^3} + \bar{\gamma}$$

The Keplerian elements become then time dependent according to the Gauss' equations. These are called *osculating elements*.



Transformation : cartesian coordinates → keplerian elements

kinetic momentum $\bar{C} = \bar{r} \times \dot{\bar{r}}$



$$\begin{cases} C = r^2\dot{\theta} = na^2\sqrt{1-e^2} = \sqrt{\mu a(1-e^2)}, \textit{areal constant} \\ \|\dot{\bar{r}}\|^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right), \textit{vis-viva equation} \end{cases}$$

$$\begin{cases} \frac{1}{a} = \frac{2}{r} - \frac{\|\dot{\bar{r}}\|^2}{\mu} \\ e^2 = 1 - \frac{C^2}{\mu a} \end{cases}$$

$$\mu = GM = n^2a^3$$

$$\begin{cases} r = a(1 - e\cos E) \\ \dot{r} = \frac{na^2}{r} e\sin E = \frac{\sqrt{\mu a}}{r} e\sin E \end{cases}$$

$$\begin{cases} \cos E = \frac{a-r}{ae} \\ \sin E = \frac{r.\dot{r}}{e\sqrt{\mu a}} \end{cases}$$

$$\begin{cases} \cos i = C_z / C \\ \sin i = \sqrt{C_x^2 + C_y^2} / C \end{cases}$$

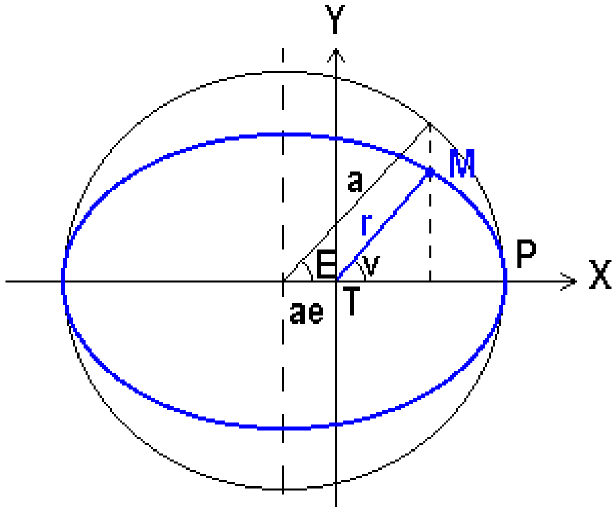
$$\begin{cases} \cos\Omega = -C_y \Big/ \sqrt{C_x^2 + C_y^2} \\ \sin\Omega = C_x \Big/ \sqrt{C_x^2 + C_y^2} \end{cases}$$

$$\begin{cases} \cos v = \frac{a}{r}(\cos E - e) \\ \sin v = \frac{a}{r}\sqrt{1-e^2}\sin E \end{cases}$$

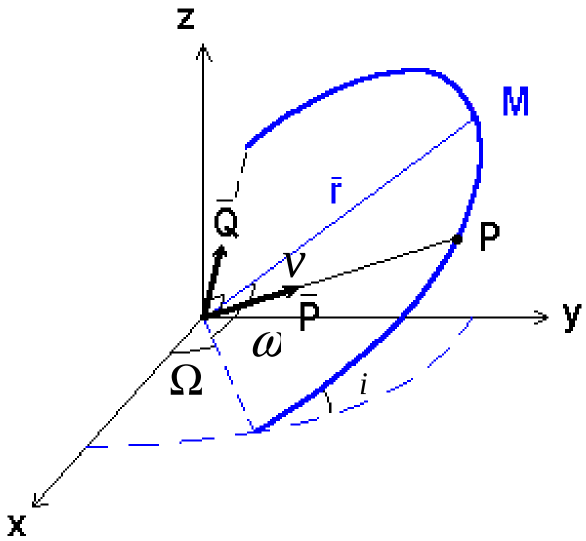
$$\begin{cases} \cos(\omega + v) = (yC_x - xC_y) \Big/ \left(r\sqrt{C_x^2 + C_y^2}\right) \\ \sin(\omega + v) = zC \Big/ \left(r\sqrt{C_x^2 + C_y^2}\right) \end{cases}$$

$$M = E - e\sin E;\ tg\frac{v}{2} = \sqrt{\frac{1+e}{1-e}}\ tg\frac{E}{2}$$

Transformation : keplerian elements → cartesian coordinates



$$\boxed{\begin{aligned}\bar{r} &= \bar{P}\,X + \bar{Q}\,Y\\ \dot{\bar{r}} &= \bar{P}\,\dot{X} + \bar{Q}\,\dot{Y}\end{aligned}}$$



$$\begin{cases} X = r\cos v = a(\cos E - e)\\ Y = r\sin v = a\sqrt{1-e^2}\,\sin E\end{cases}$$

$$\begin{cases} \bar{r} = \Re_3(-\Omega)\,\Re_1(-i)\,\Re_3(-\omega)\,(X, Y, 0)^T\\ \dot{\bar{r}} = \Re_3(-\Omega)\,\Re_1(-i)\,\Re_3(-\omega)\,(\dot{X}, \dot{Y}, 0)^T\end{cases}$$

$$\begin{cases} \dot{X} = \dot{r}\cos v - r\dfrac{dv}{dt}\sin v = -a\dfrac{dE}{dt}\sin E\\ \dot{Y} = \dot{r}\sin v + r\dfrac{dv}{dt}\cos v = a\sqrt{1-e^2}\,\dfrac{dE}{dt}\cos E\end{cases}$$

$$\bar{P}\begin{pmatrix}\cos\Omega\cos\omega - \cos i\sin\Omega\sin\omega\\ \sin\Omega\cos\omega + \cos i\cos\Omega\sin\omega\\ \sin i\sin\omega\end{pmatrix}$$

$$\dot{r} = \frac{nae}{\sqrt{1-e^2}}\sin v\,,\ \dot{\theta} = \frac{C}{r^2} = \frac{na^2\sqrt{1-e^2}}{r^2}\,,\ \frac{dE}{dt} = \frac{na}{r}$$

$$\bar{Q} = \frac{\partial\bar{P}}{\partial\omega}\begin{pmatrix}-\cos\Omega\sin\omega - \cos i\sin\Omega\cos\omega\\ -\sin\Omega\sin\omega + \cos i\cos\Omega\cos\omega\\ \sin i\cos\omega\end{pmatrix}$$

Numerical vs. analytical methods

Numerical

• Numerical integration (of second order) of the fundamental equation of dynamics in Cartesian coordinates:

$$r = \iint \ddot{\bar{r}}\,dt \quad ; \quad \ddot{\bar{r}} = \sum_n \bar{A}(\bar{r}, \dot{\bar{r}}, \alpha_i)$$

from a set of initial conditions at t_0 (orbit and acceleration model parameters):

$$\bar{r}_0, \dot{\bar{r}}_0, \alpha_i = (C_{lm}, S_{lm}, \ldots, p_{dyn.})$$

• Adjustment of initial orbit parameters as well as of model parameters according to tracking observations:

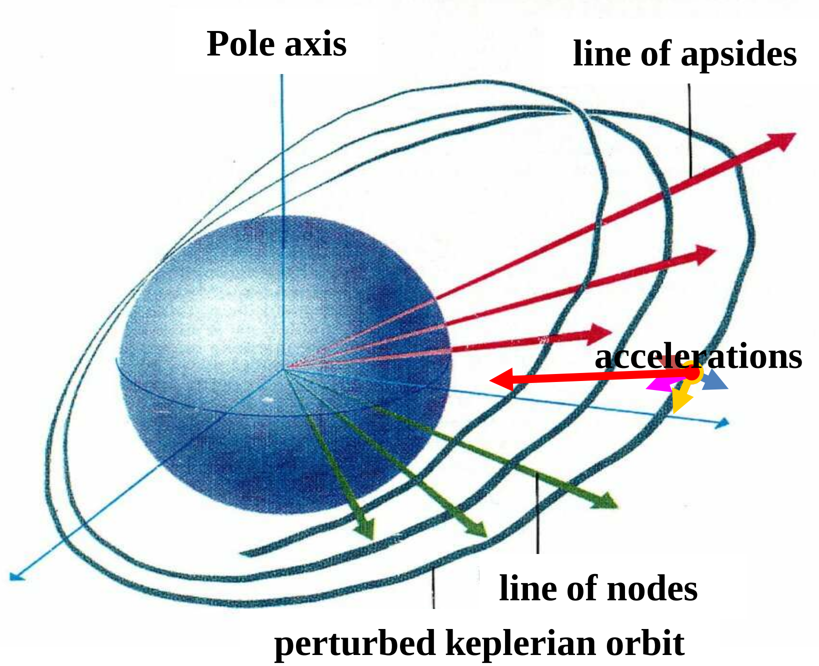
$$\Delta Q = Q_{obs} - Q_{calc} = \frac{\partial Q}{\partial \bar{r}_0}\Delta \bar{r}_0 + \frac{\partial Q}{\partial \dot{\bar{r}}_0}\Delta \dot{\bar{r}}_0 + \sum_i \frac{\partial Q}{\partial \alpha_i}\Delta \alpha_i$$

• Iterative method

Analytical

• Integration (of first order) of Gauss / Lagrange's equations in Keplerian elements

$$\left(\frac{da}{dt}, \frac{de}{dt}, \frac{di}{dt}, \frac{d\Omega}{dt}, \frac{d\omega}{dt}, \frac{dM}{dt}\right)$$

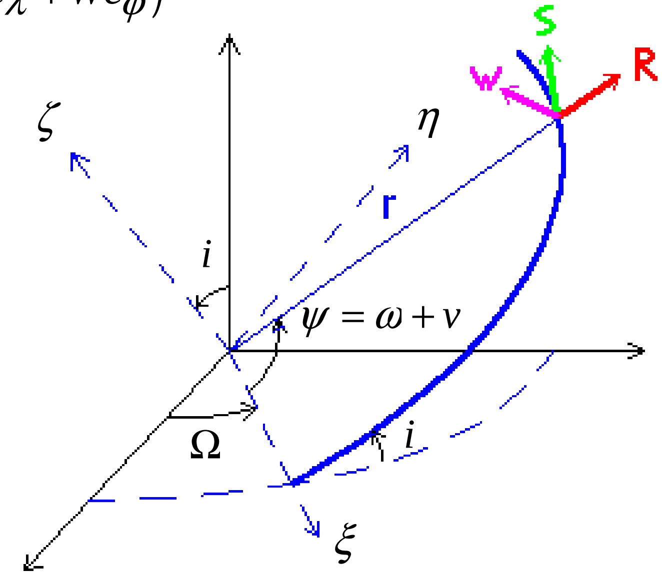


Gauss' equations

Johann Carl Friedrich Gauss
1777-1855

They are based on the energy relationship and on the theorem of kinetic energy: *the work done on an object by a net force equals the change in kinetic energy of the object* :

$$d\left(\frac{1}{2}mv^2\right)=dW \quad \left(\text{differential work with} : d\bar{F}=R\bar{e}_r+S\bar{e}_\lambda+W\bar{e}_\varphi\right)$$



$$\left[\begin{aligned}
\frac{da}{dt}&=\frac{2}{n\sqrt{1-e^2}}\left[Re\sin v+S(1+e\cos v)\right]\\
\frac{de}{dt}&=\frac{\sqrt{1-e^2}}{na}\left[R\sin v+S(\cos E+\cos v)\right]\\
\frac{di}{dt}&=-\frac{Wr\cos(\omega+v)}{na^2\sqrt{1-e^2}}\\
\frac{d\Omega}{dt}&=-\frac{Wr\sin(\omega+v)}{na^2\sqrt{1-e^2}\sin i}\\
\frac{d\omega}{dt}&=\frac{\sqrt{1-e^2}}{nae}\left[-R\cos v+S\cos v\left(1+\frac{1}{1+e\cos v}\right)-\frac{Wr\cos i\sin(\omega+v)}{na^2\sqrt{1-e^2}\sin i}\right]\\
\frac{dM}{dt}&=n-\frac{1-e^2}{nae}\left[R\left(\cos v-\frac{2e}{1+e\cos v}\right)-S\sin v\left(1+\frac{1}{1+e\cos v}\right)\right]
\end{aligned}\right.$$

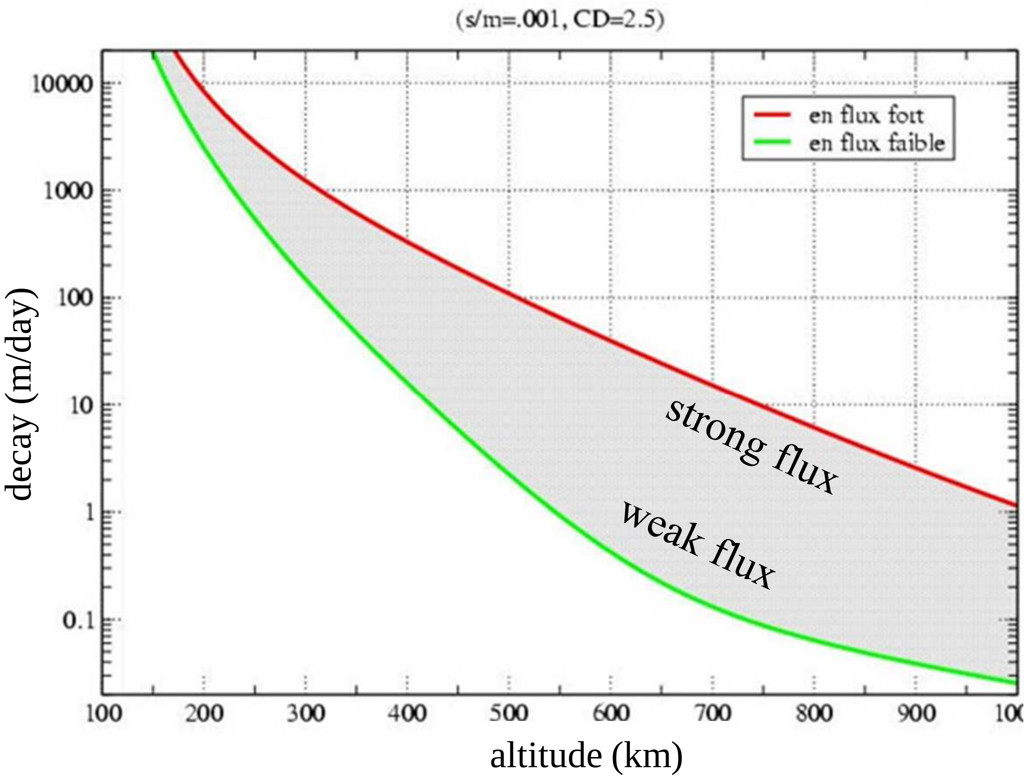
$$\text{with} \quad n=\sqrt{\frac{GM}{a^3}}$$

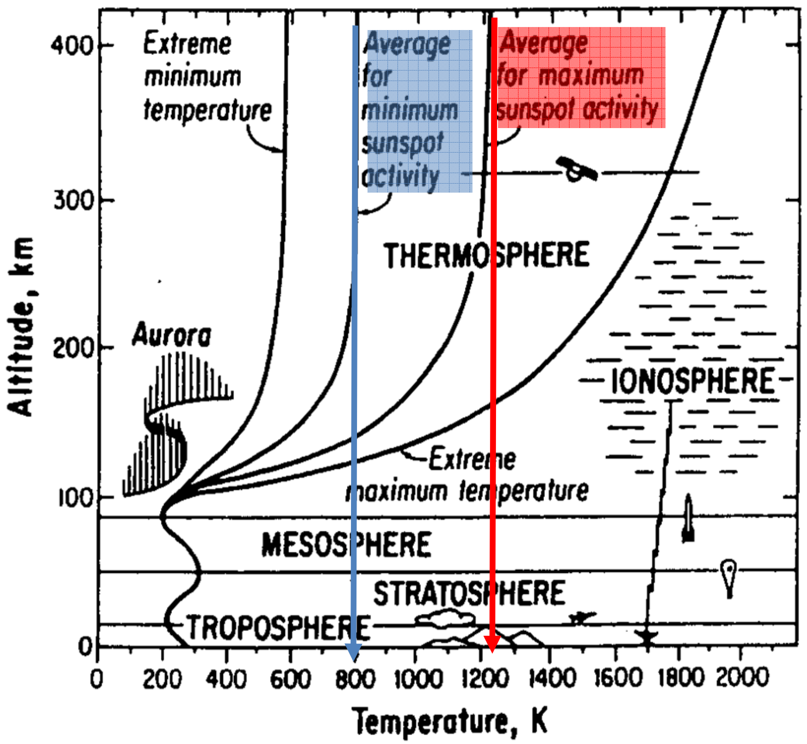
Orbit decay

Particular case of atmospheric drag: decay of the semi-major axis (circular orbit):

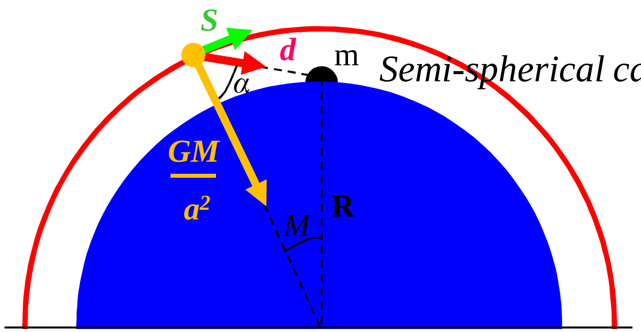
$$\frac{da}{dt} = \frac{2}{n\sqrt{1-e^2}}\left[Re \sin v + S\left(1 + e \cos v\right)\right] \approx \frac{2\,S}{n} = \frac{-\rho\, C_D\, \frac{s}{m}\, v_r^2}{n} \approx -\rho\, C_D\, \frac{s}{m}\sqrt{a\,GM}$$

Orbit decay wrt altitude (S/m=.001, CD=2.5)





Effect of a superficial mass on a circular orbit at 400 km altitude



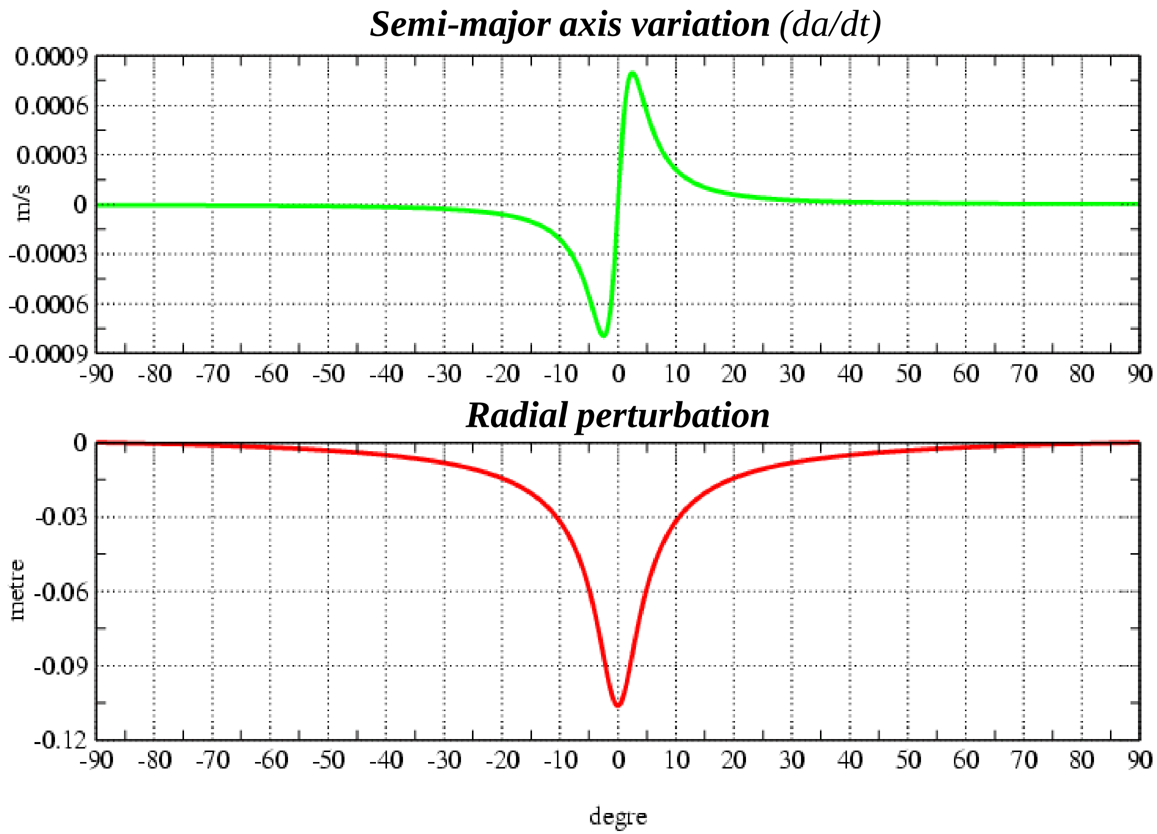
Semi-spherical cap with a 8 km height (density: 2,67 t/m³)

Additional newtonian gravitational acceleration projected onto the tangent to the orbit:

$$S = \frac{Gm}{d^2} \sin\alpha = \frac{R\,Gm}{d^3} \sin nt$$

The altitude variation of the orbit is proportional to the generated acceleration, perpendicular to the ray vector: $\dfrac{da}{dt} = \dfrac{2\,S}{n}$

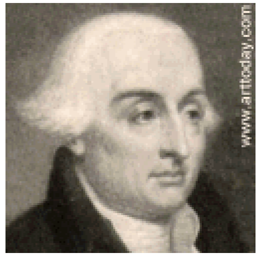
The integrated radial perturbation reaches 10 cm at the zenith of the mass.



Lagrange's equations

When accelerations derive from a potential: $\bar{\gamma} = \overline{grad}\, U$ (case of gravitation), this potential can be differentiated with respect to the keplerian orbital elements by linear combination of the perturbing acceleration components R, S, W in the orbital frame :

$$\frac{\partial U}{\partial \alpha} = R\frac{\partial r}{\partial \alpha} + Sr\frac{\partial \psi}{\partial \alpha} + W\frac{\partial \zeta}{\partial \alpha} \quad (\alpha = a, e, i, \omega, \Omega, M)$$



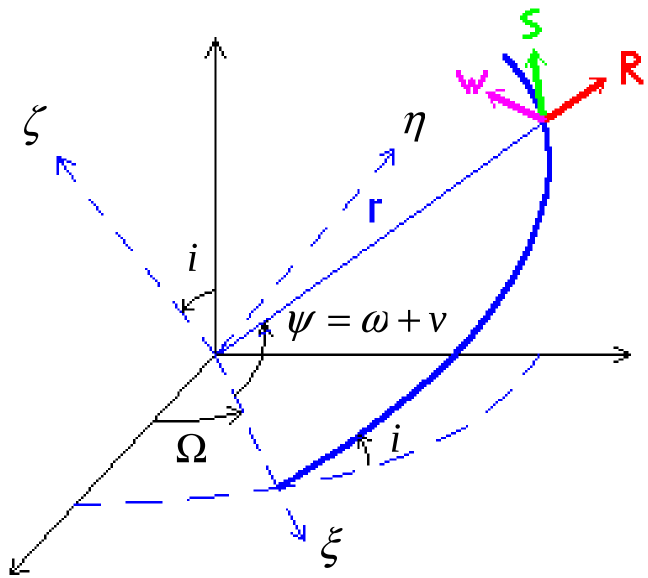
Joseph Louis Lagrange
1736-1813

metric elements

$$\begin{cases} \frac{da}{dt} = \frac{2}{n.a}\frac{\partial U}{\partial M} \\ \frac{de}{dt} = -\frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial U}{\partial \omega} + \frac{1-e^2}{na^2 e}\frac{\partial U}{\partial M} \\ \frac{di}{dt} = -\frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial U}{\partial \Omega} + \frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial U}{\partial \omega} \end{cases}$$

angular elements

$$\begin{cases} \frac{d\Omega}{dt} = \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial U}{\partial i} \\ \frac{d\omega}{dt} = \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial U}{\partial e} - \frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial U}{\partial i} \\ \frac{dM}{dt} = n - \frac{2}{na}\frac{\partial U}{\partial a} - \frac{1-e^2}{na^2 e}\frac{\partial U}{\partial e} \end{cases}$$



Gravitational potential modeling

According to the Newton's law of gravitation : $U = G\iiint_V \frac{dm}{\Delta} = \frac{G}{r}\iiint_V \frac{r}{\Delta}\, dm$

Adrien-Marie Legendre
1752 - 1833

$V = r/\Delta$ can be expanded in base of spherical harmonic functions when r'< r:

$$V = \sum_{n=0}^{\infty} \rho^n P_n \quad with \quad \rho = \frac{r'}{r} < 1$$

$$and \quad P_n(\cos\theta) = \sum_{m=0}^{n} (2-\delta_{m0}) \frac{(n-m)!}{(n+m)!} P_{n,m}(\sin\varphi) P_{n,m}(\sin\varphi') \cos(m(\lambda-\lambda'))$$

(Legendre functions)

hence :

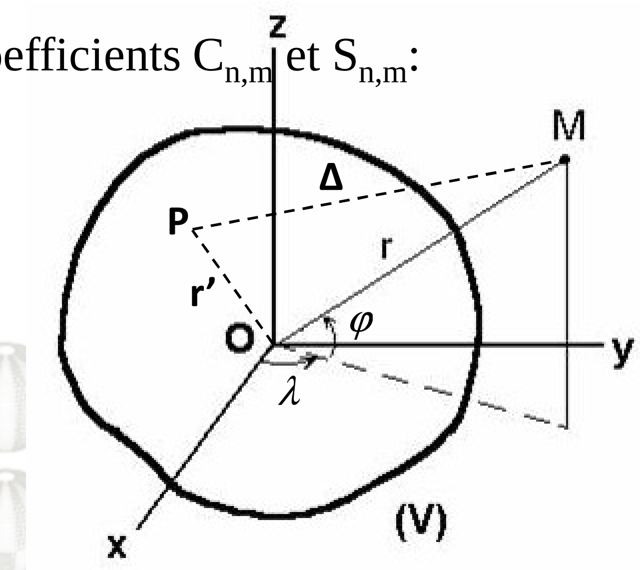
$$U = \frac{G}{r}\iiint_V \sum_{n=0}^{\infty}\sum_{m=0}^{n} (2-\delta_{m0}) \frac{(n-m)!}{(n+m)!} \left(\frac{r'}{r}\right)^n P_{n,m}(\sin\varphi) P_{n,m}(\sin\varphi') \cos(m(\lambda-\lambda'))\, dm$$

or when isolating the integral part under the form of Stokes' coefficients $C_{n,m}$ et $S_{n,m}$:

$$U = \frac{GM}{a_e} \sum_{n=0}^{\infty} \left(\frac{a_e}{r}\right)^{n+1} \sum_{m=0}^{n} P_{n,m}(\sin\varphi)\left(C_{n,m}\cos m\lambda + S_{n,m}\sin m\lambda\right)$$

with :

$$M a_e^n \begin{Bmatrix} C_{n,m} \\ S_{n,m} \end{Bmatrix} = \iiint_V (2-\delta_{m0}) \frac{(n-m)!}{(n+m)!} r'^n P_{n,m}(\sin\varphi') \begin{Bmatrix} \cos m\lambda' \\ \sin m\lambda' \end{Bmatrix} dm$$



a_e = 6378 136.46 m
GM = 398 600.4415 km^3/s^2

Kaula's expansion (1966)

William M. Kaula
1926-2000

Expanding the potential at the point (r, φ, $\lambda + \theta$) in the instantaneous reference frame:

$$U = \frac{GM}{R}\sum_{l=2}^{\infty}\sum_{m=o}^{l}\left(\frac{R}{r}\right)^{l+1}\left(C_{lm}\cos m(\lambda+\theta)+S_{lm}\sin m(\lambda+\theta)\right)P_{lm}(\sin\varphi)$$

$$= \frac{GM}{R}\sum_{l=2}^{\infty}\sum_{m=o}^{l}\left(\frac{R}{r}\right)^{l+1}R_e\left[(C_{lm}-iS_{lm})e^{im(\lambda+\theta)}\right]P_{lm}(\sin\varphi) \qquad (\theta\text{: sidereal time})$$

Transformation of polar coordinates into Keplerian orbital parameters:

$$\left(\frac{1}{r}\right)^{l+1}P_{lm}(\sin\varphi)e^{im(\lambda+\theta)} = \sum_{p=o}^{l}\sum_{q=-\infty}^{+\infty}\left(\frac{1}{a}\right)^{l+1}F_{lmp}(i)\,G_{lpq}(e)\left[\begin{pmatrix}\cos\psi_{lmpq}\\ \sin\psi_{lmpq}\end{pmatrix}+i\begin{pmatrix}\sin\psi_{lmpq}\\ -\cos\psi_{lmpq}\end{pmatrix}\right] \quad \begin{matrix}l-m\ \text{even}\\ l-m\ \text{odd}\end{matrix}$$

with $\psi_{lmpq} = (l-2p)\omega + (l-2p+q)M + m(\Omega-\theta)$

Kaula's expansion:

$$U = \sum_{l=2}^{\infty}\sum_{m=o}^{l}\sum_{p=o}^{l}\sum_{q=-\infty}^{+\infty}U_{lmpq}$$

with $U_{lmpq} = \frac{GM}{R}\left(\frac{R}{a}\right)^{l+1}F_{lmp}(i)\,G_{lpq}(e)\,S_{lmpq}(\omega,\Omega,M,\theta)$

and $S_{lmpq} = \begin{pmatrix}C_{lm}\\ -S_{lm}\end{pmatrix}\cos\psi_{lmpq} + \begin{pmatrix}S_{lm}\\ C_{lm}\end{pmatrix}\sin\psi_{lmpq} \quad \begin{matrix}l-m\ \text{even}\\ l-m\ \text{odd}\end{matrix}$

Potential of degree 2: $U_2 = \sum_{p=o}^{2} \sum_{q=-\infty}^{+\infty} U_{20pq}$

$$U_2 = \frac{GM}{R}\left(\frac{R}{a}\right)^3 \sum_{p=o}^{2} \sum_{q=-1}^{1} F_{2op}(i)\,G_{2pq}(e)\,C_{20}\cos\left[(2-2p)(\omega+M)+qM\right]$$

for $e << 1$, one just keeps $q = o$ ($G(e)\ \alpha\ e^{|q|}$):

p = 0 : $\psi_{2000} = 2(\omega + M)$

p = 1 : $U_{2010} = \frac{GM}{R}\left(\frac{R}{a}\right)^3 F_{201}(i)\,G_{210}(e)\,C_{20}$ with $F_{201}(i) = \frac{3}{4}\sin^2 i - \frac{1}{2}$

p = 2 : $\psi_{2020} = -2(\omega + M)$ and $G_{210}(e) = \left(1-e^2\right)^{-\frac{3}{2}}$

Lagrange's equation ⇒ secular perturbations

$$\begin{cases} \dfrac{da}{dt} = \dfrac{de}{dt} = \dfrac{di}{dt} = 0 \\[2ex] \dfrac{d\Omega}{dt} = \dot{\Omega} = \dfrac{3n}{2\left(1-e^2\right)^2}\left(\dfrac{R}{a}\right)^2 C_{20}\cos i \\[2ex] \dfrac{d\omega}{dt} = \dot{\omega} = \dfrac{3n}{4\left(1-e^2\right)^2}\left(\dfrac{R}{a}\right)^2 C_{20}\left(1-5\cos^2 i\right) \\[2ex] \dfrac{dM}{dt} = \dot{M} = n + \dfrac{3n}{4\left(1-e^2\right)^{3/2}}\left(\dfrac{R}{a}\right)^2 C_{20}\left(1-3\cos^2 i\right) \end{cases}$$

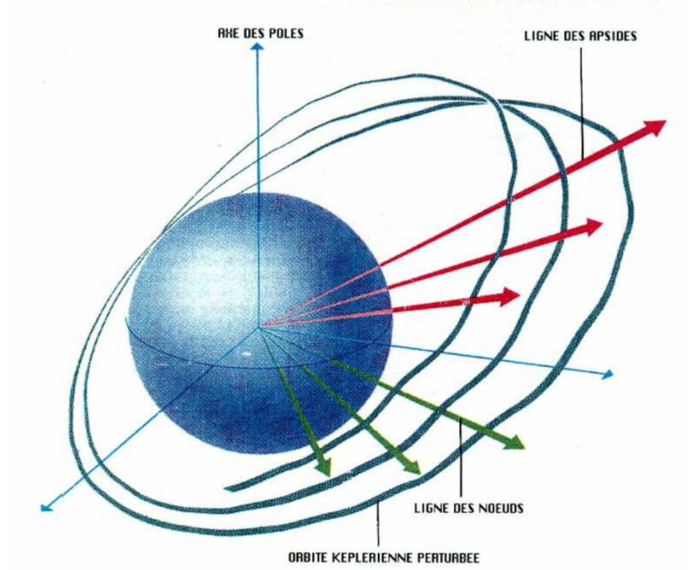
$$\left(n = \sqrt{\frac{GM}{a^3}}\right)$$

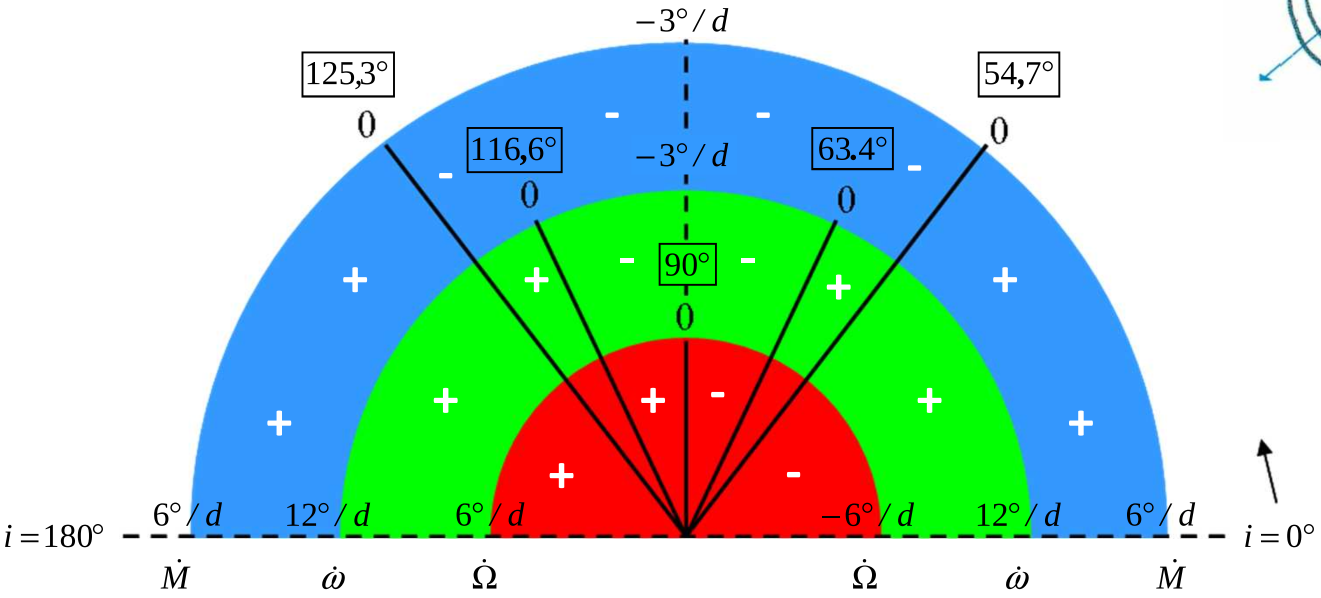
Secular perturbations from C_{20}

Ascending node : $$\dot{\Omega} = \frac{3}{2} n C_{20} \left(\frac{a_e}{a}\right)^2 \frac{1}{\left(1-e^2\right)^2} \cos i$$

Perigee : $$\dot{\omega} = \frac{3}{4} n C_{20} \left(\frac{a_e}{a}\right)^2 \frac{1}{\left(1-e^2\right)^2} \left(1-5\cos^2 i\right)$$

Mean motion : $$\dot{M} = n + \frac{3}{4} n C_{20} \left(\frac{a_e}{a}\right)^2 \frac{1}{\left(1-e^2\right)^{3/2}} \left(1-3\cos^2 i\right)$$





$$e \uparrow \Rightarrow \dot{\Omega},\ \dot{\omega},\ \dot{M} \uparrow$$

$$a \uparrow \Rightarrow \dot{\Omega},\ \dot{\omega},\ \dot{M} \downarrow$$

variations of secular perturbations function of inclination for a satellite at 1000 km (e = 0)

Orbital secular drifts from C_{20} coefficient

Satellite	*h (km)*	*i (deg)*	*n (rev/day)*	*Period (min)*	*dΩ/dt (deg/day)*	*dω/dt (deg/day)*
GOCE	260	96.5	16.1	90	0.99	-2.96
GRACE	470	89	15.2	94	-0.13	-3.84
SPOT	830	98.7	14.2	101	0.99 (365d)	-2.89 (125d)
Jason	1335	66	12.8	112	-2.08 (173d)	-0.45 (801d)
LAGEOS	5900	110	6.4	226	0.34	-0.22
GPS	20000	55	2.0	11.9 hr	-0.04	0.02
Galileo	23200	56	1.7	14.1 hr	-0.03	0.02
Meteosat	35780	7	1.0	23.9 hr	-0.01	0.03

Geosynchronous condition

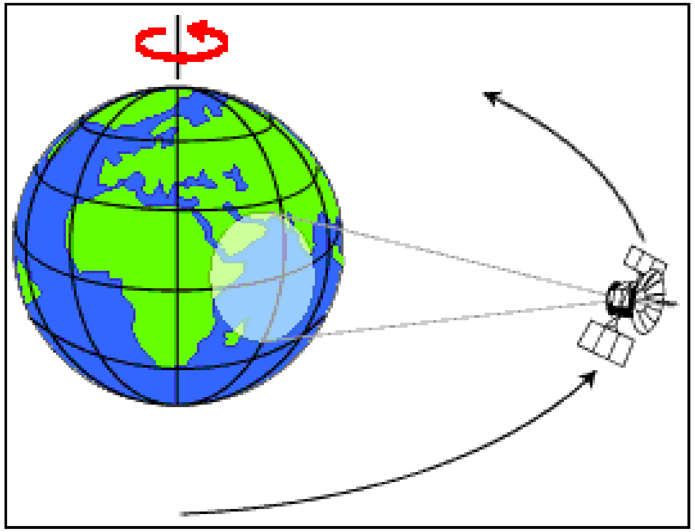
The revolution period equals the sidereal period of the Earth's rotation (one revolution in 86164 s) :

$$\left| \begin{array}{l} e = 0 \\ i = 0 \\ \dot{\omega} + \dot{\Omega} + \dot{M} = \dot{\Omega}_E = 7{,}292115\,10^{-5}\ rad/s = 360.9856\ deg/day \quad (2\pi/86164.1\ s) \end{array} \right.$$

=> a = **42166 km** and h = **35795 km**

First Geosynchronous satellites:

- Syncom-2 (communication), i=32.8 deg., 1963…
- SMS (meteorology), i=15.5 deg., 1974…
- Kosmos-637, 1974…
- METEOSAT, 1977…



Drift in longitude:

$$n^2 a^3 = GM \quad \Rightarrow \quad \frac{dn}{n} = -\frac{3}{2}\frac{da}{a} \quad \Rightarrow \quad \Delta n = -\frac{3}{2}\frac{\Delta a}{a_{Geo}}\dot{\Omega}_E \approx -1.43\,\Delta a \quad (m/day)$$

Drift in latitude: Bernouilli's lemniscate

A geosynchronous orbit is not completely stable. Parameters experience changes, like:

- a, by the effect of some tesseral terms of the Earth's gravity field.
- e, by the effect of radiation pressure;
- i, by the effect of Moon and Sun gravitational attraction.

The (2,2) tesseral terms of the gravity field generate 2 stable points (75.1 E and 105.3 W) toward which satellites migrate and 2 unstable points (164 E and 11 W) from which satellites move away.

Due to these effects, manoeuvres are mandatory for maintaining geosynchronism. The classical acceptable shift is $\pm$1 deg in East-West and $\pm$0.1 deg in North-South directions.

Geosynchronous satellites which should observe northern areas (Russia, Canada) are put mostly at 63.4 deg. inclination to freeze the perigee (e.g. Tundra: h_a = 47034 km, h_p = 24536 km, e = 0.2668 and Supertundra: h_a = 53020 km, h_p = 17950 km, e = 0.4230). However several satellites are necessary to get a full visibility at any time.

Sunsynchronous condition

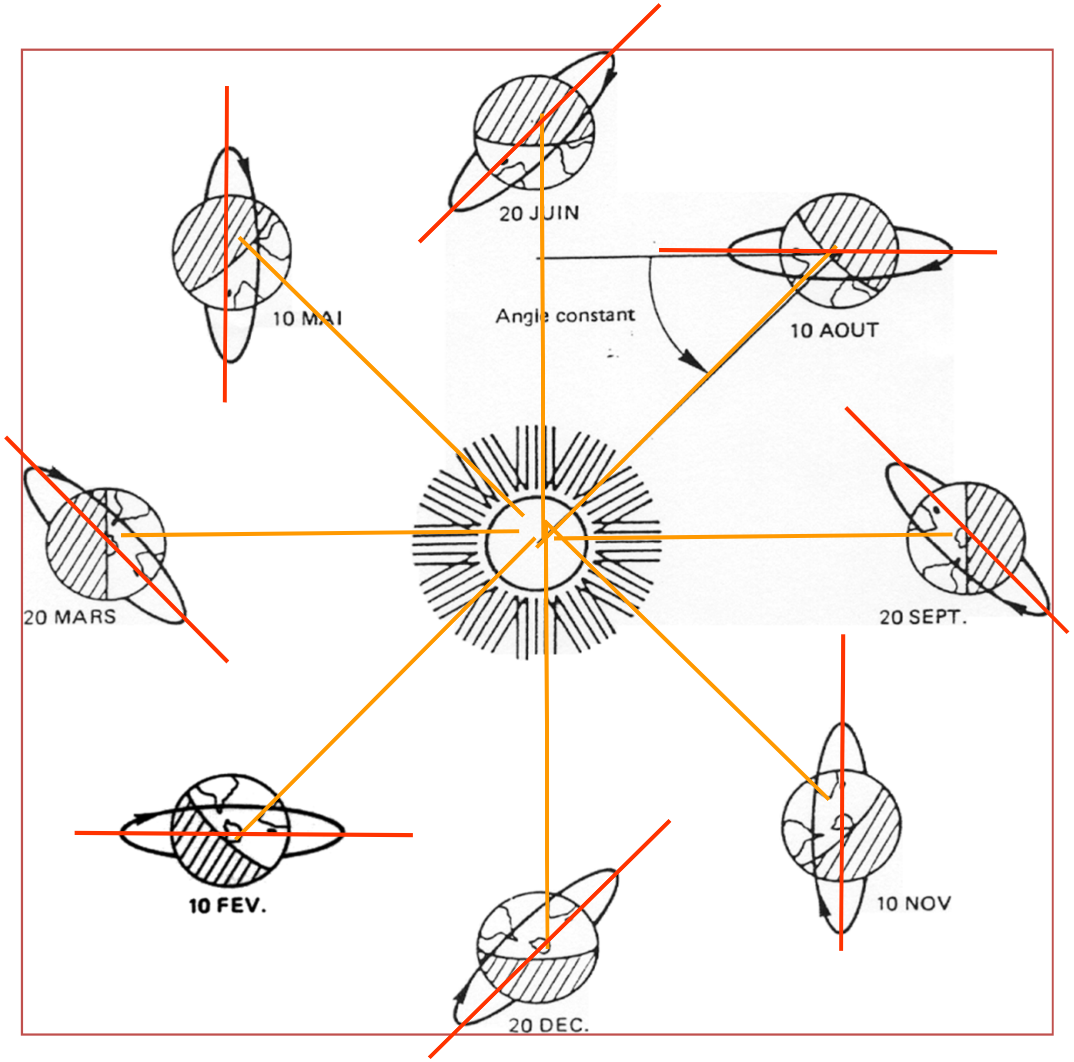
The node line follows the (mean) Sun motion (one revolution in a year):

$$\dot{\Omega} = \frac{3}{2} n \frac{R^2}{a^2} C_{20} \frac{1}{(1-e^2)^2} \cos i = \dot{\Omega}_s$$

Like the Earth, the orbital plane accomplishes one revolution around the Sun in one tropical year:

$$\dot{\Omega}_S = \frac{2\pi}{T_{trop}} = \frac{360\ deg}{365.2422\ day}$$

$$\boxed{\dot{\Omega}_S = 0.9856\ deg/day}$$



Sunsynchronous condition

The sunsynchronous condition can only be realized for retrograde inclinations:

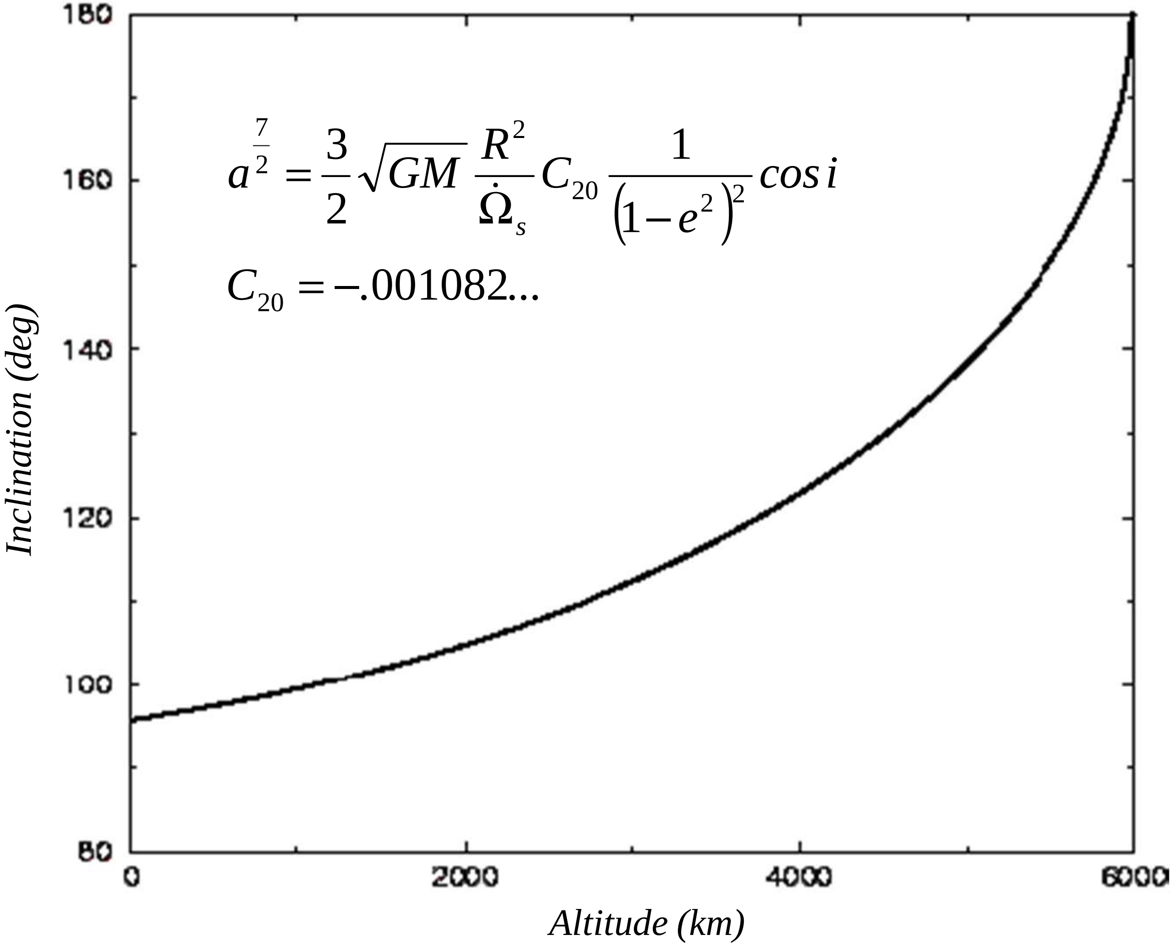
95.7 deg < i < 180 deg

and altitudes less than:

h = 5 960 km

(⇔ a < 12 331 km)

Inclination wrt altitude for circular orbits



$$a^{\frac{7}{2}} = \frac{3}{2}\sqrt{GM}\,\frac{R^2}{\dot{\Omega}_s}C_{20}\frac{1}{\left(1-e^2\right)^2}\cos i$$

$$C_{20} = -.001082...$$

First sunsynchronous satellites

• military: SAMO2-2 (Satellite and Missile Observation System), 31 january 1961, $i=97{,}4^{\circ}$, 474-557km

• meteorology: NIMBUS-1 (1964)

• Earth observation: Landsat (1972)

Orbits are defined by their local time at the ascending and descending nodes

SPOT4: $\tau_{DN} = 10{:}30 \pm 00{:}10$ within $\pm$ 3 km

Maneuvers are required each 2 to 8 weeks because of 3 particular effects on the ascending (or descending) node:

- irregular variations of the true Sun longitude (indirect effect on Ω)
- atmospheric drag (effect on a)
- gravitational perturbation of the Sun (effect on i)

Periodic perturbations from C_{20}

$$U_2 = \frac{GM}{R}\left(\frac{R}{a}\right)^3 \sum_{p=o}^{2}\sum_{q=-1}^{1} F_{2op}(i)G_{2pq}(e)C_{20}\cos[(2-2p)(\omega+M)+qM]$$

In second iteration, considering that only ψ_{lmpq} is time depending :

p = 0: $\psi_{2020} = 2(\overline{\omega}+\overline{M})$ ($\overline{\omega}$ *and* $\overline{M}$ *time dependent from secular effects*)

p = 1 : $U_{2010} = \frac{GM}{R}\left(\frac{R}{a}\right)^3 F_{201}(i)G_{210}(e)C_{20}$

p = 2: $\psi_{2020} = -2(\overline{\omega}+\overline{M})$

with $F_{200}(i) = F_{202}(i) = -\frac{3}{8}\sin^2 i$

and $G_{200}(e) = G_{220}(e) = 1-\frac{5}{2}e^2$

Lagrange's equation ⇒ periodic perturbations

$$\left(\frac{da}{dt}\right)_{2opo} = \frac{2}{na}\frac{\partial U_{2opo}}{\partial M}$$

$$\frac{\partial U_{2opo}}{\partial M} = -GM\frac{R^2}{a^3}(2-2p)F_{2op}(i)G_{2po}(e)C_{20}\sin[(2-2p)(\overline{\omega}+\overline{M})]$$

and after integration :

$$\Delta a_{2opo} = 2\frac{GM}{na^2}\frac{R^2}{a^2}\frac{(2-2p)F_{2op}(i)G_{2po}(e)C_{20}\cos[(2-2p)(\overline{\omega}+\overline{M})]}{(2-2p)(\dot{\overline{\omega}}+\dot{\overline{M}})}$$

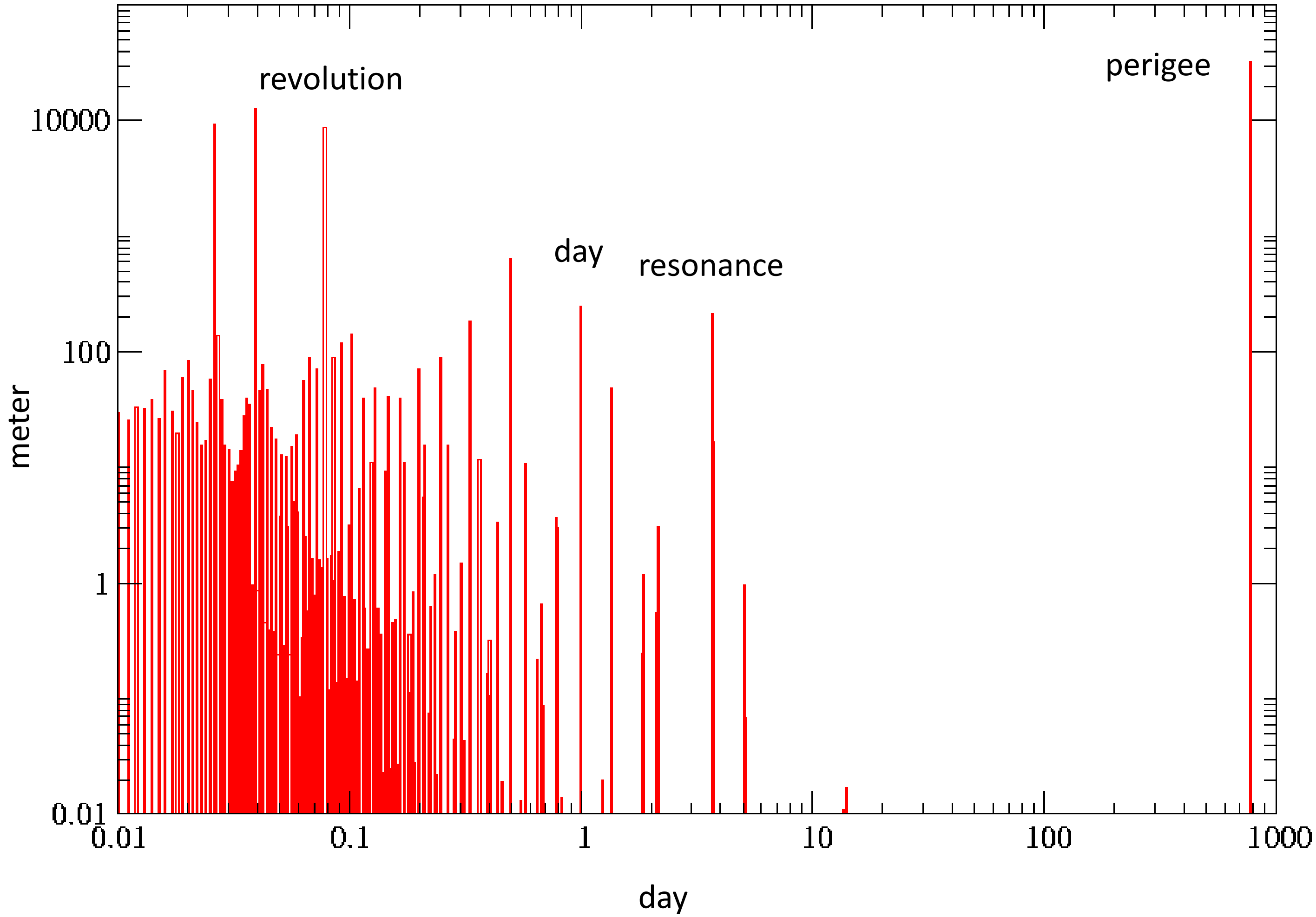
Perturbation spectrum

Harmonics	Index			Frequency	Period	Element
$\dot{\psi}=$		$+(l-2p)\dot{\omega}$	$+(l-2p+q)\dot{M}$	$\dot{\psi}$		
Even zonal	0	0	0	*0*	secular	ω, Ω, M
Even zonal	0	0	≠0	$q\dot{M}$	revolution	$a, e, i,$ ω, Ω, M
Odd zonal (or even 2nd order)	0	≠0	0	$(l-2p)\dot{\omega}$	perigee	$e, i,$ ω, Ω, M
Tesseral	≠0	0	0		day	$i,$ ω, Ω, M
Tesseral of resonance	14* 28*	1 2	1 2	$(\dot{\omega}+\dot{M})-m(\dot{\theta}-\dot{\Omega})$ $2(\dot{\omega}+\dot{M})-2m(\dot{\theta}-\dot{\Omega})$	> day and sub-multiples	$e, i,$ ω, Ω, M

* *for a ~800 km altitude*

Perturbation spectrum

Jason orbit



Frozen orbit

Sometimes it can be interesting to have a frozen configuration of the orbit altitude wrt the Earth surface: that's the frozen orbit which is defined by a static perigee. It happens at the so-called critical inclinations when considering the C_{20} linear perturbation:

$$\boxed{\dot{\omega}_{20} = \frac{3}{4} n C_{20} \left(\frac{a_e}{a}\right)^2 \frac{1}{\left(1-e^2\right)^2}\left(1-5\cos^2 i\right) = 0 \quad \Rightarrow \quad i = 63.4\ deg\ /\ 116.6\ deg}$$

Now considering higher degree terms (like C_{30}) one can find eccentricity values which allow to get frozen orbits at different inclinations:

$$\frac{de}{dt} = -\frac{3}{8} n C_{30} \left(\frac{a_e}{a}\right)^3 \frac{\sin i \cos\omega}{\left(1-e^2\right)^2}\left(1-5\cos^2 i\right) = 0$$

$$\frac{d\omega}{dt} = \dot{\omega}_{20} - \frac{3}{8} n C_{30} \left(\frac{a_e}{a}\right)^3 \frac{\sin\omega}{e \sin i \left(1-e^2\right)^3}\begin{bmatrix} \left(1-5\cos^2 i\right)\sin^2 i \\ +e^2\left(1-\frac{35}{4}\sin^2 i \cos^2 i\right) \end{bmatrix} = 0$$

That's the case of Jason satellites which have a frozen perigee at ω = -90 deg. thanks to a well chosen eccentricity although their inclinations are 66.4 deg.

Jason-2 altitude

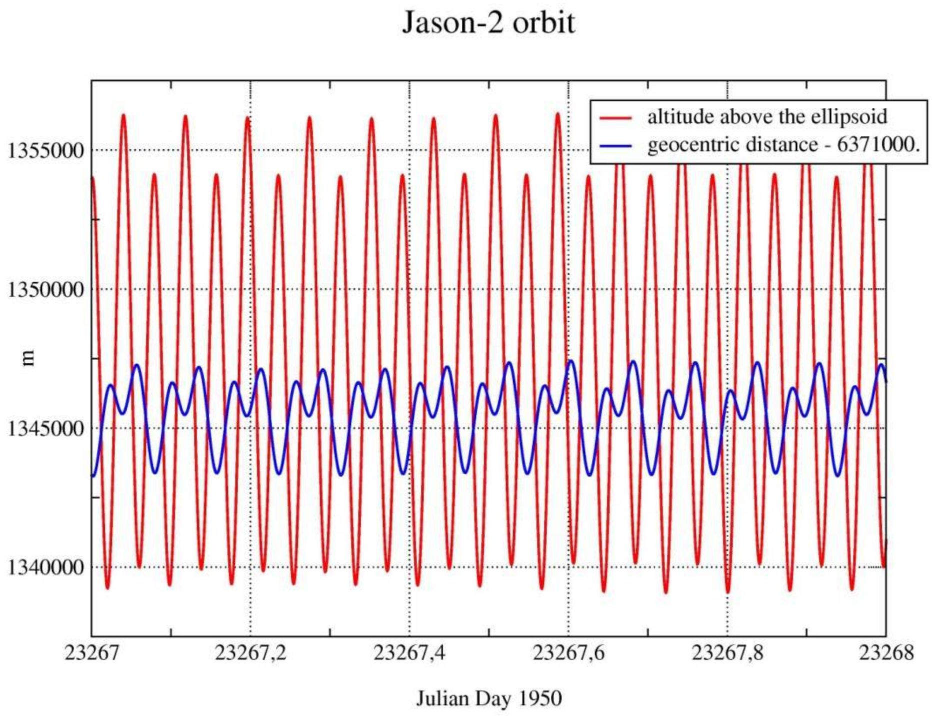
2 effects are to be considered due to :

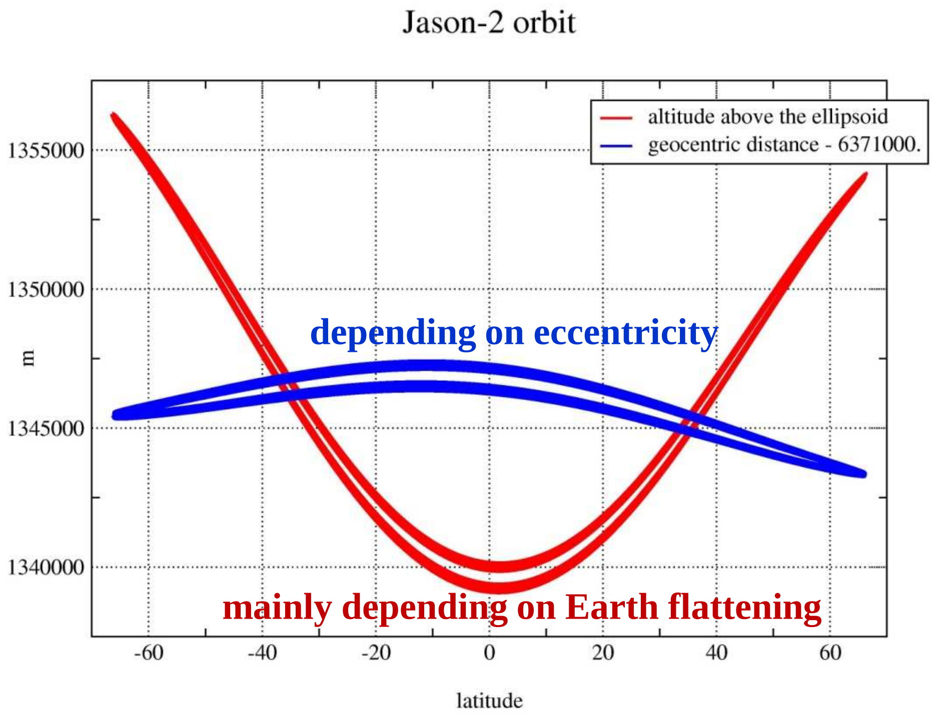
- orbit eccentricity : .0006 → 4.6 km
- Earth flattening: 1/298.257 → 21.4 km

Earth's ellipsoid parameters (IERS 2010) :
semi-major axis: 6 378 136.6 m (Topex: …6.3 m)
flattening: 1/298.25642 (Topex: 1/298.257)
→ semi-minor axis: a_e - 21 384.7 m
Mean ellipsoidal radius (Chambat & al., 2001) :

- arithmetic (GRS80): 6 371 008.8 m
- equisurfacic: 6 371 007.2 m
- equivolumetric: 6 371 000.8 m
- radial: 6 370 994.4 m

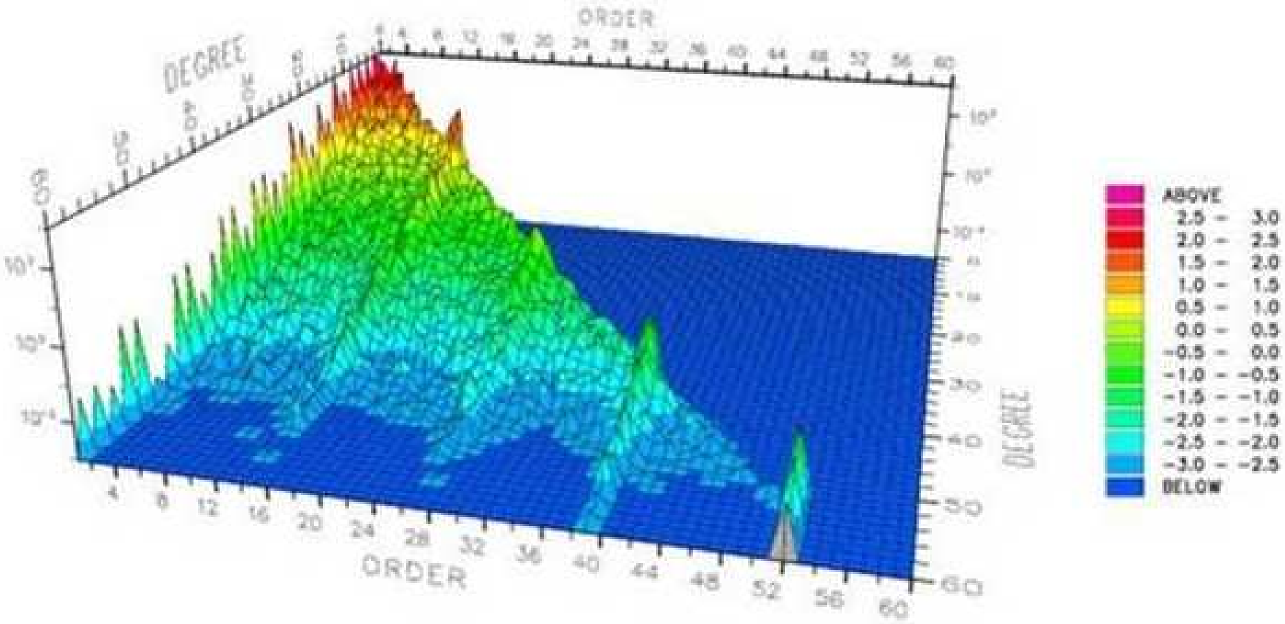
Mean topographic radius:
6 371 200 m < R < 6 371 230 m



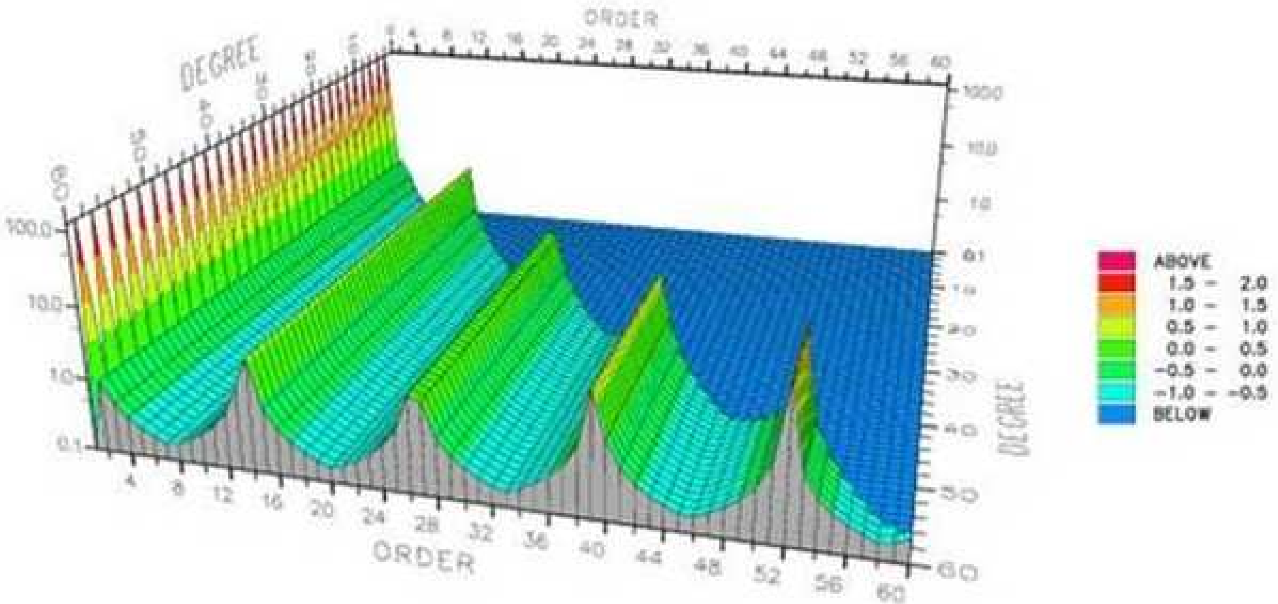


Jason
(1336 km at the equator)

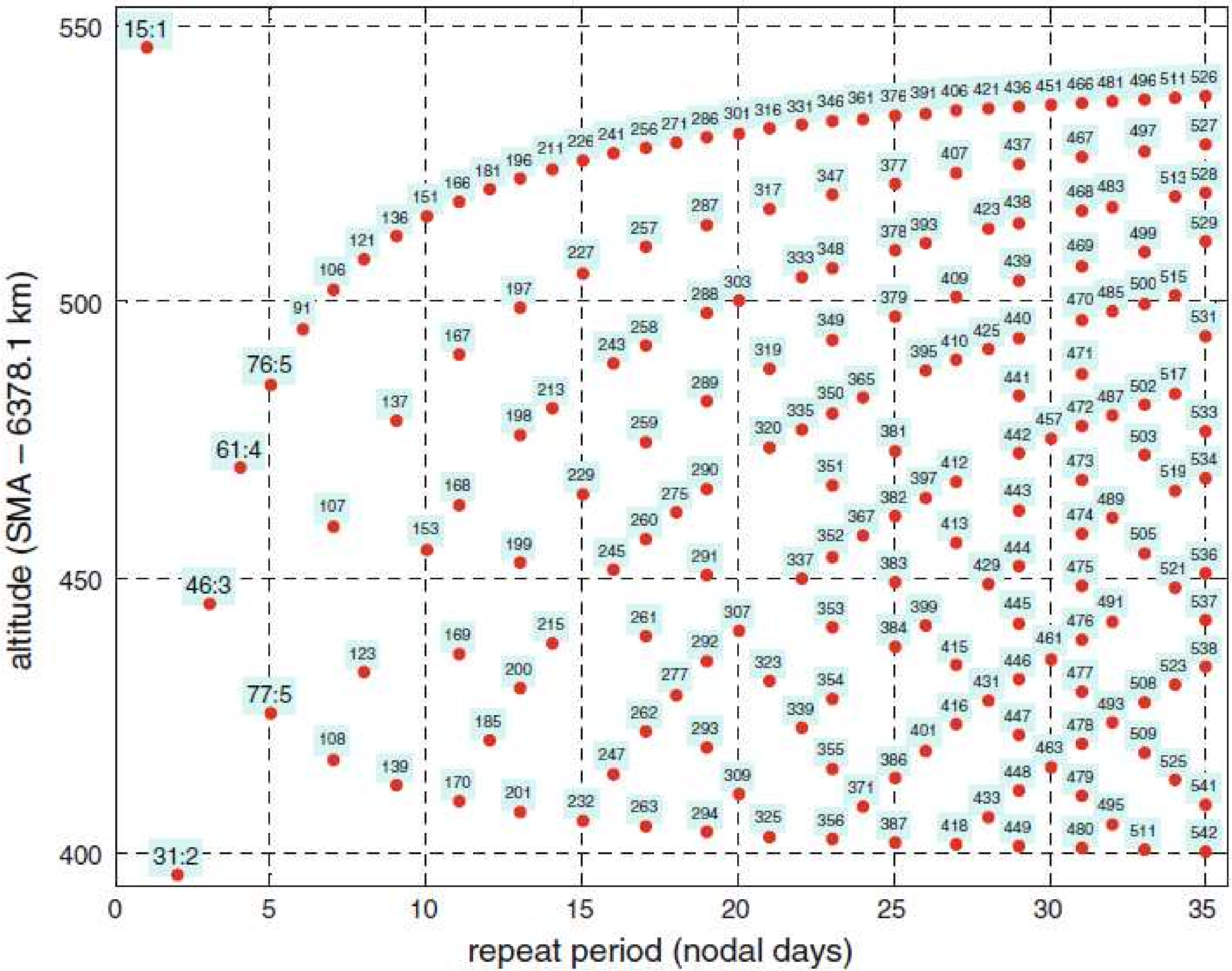
Amplitudes of orbit perturbations
from gravity field expanded in spherical harmonic functions per degree and order (in meter)



Maximal periods of orbit perturbations (in day)



Orbital resonances of GRACE (Earth, i=89.03 deg)

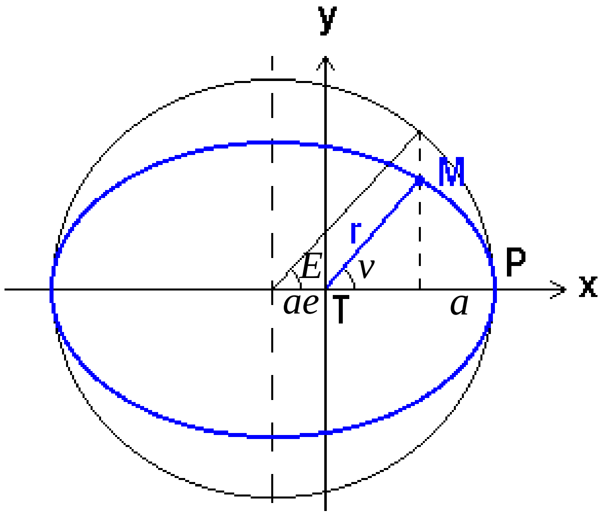


Projection in radial/transverse/normal

Radial perturbation : $r = a(1 - e\cos E)$

$$\Delta r = \frac{\partial r}{\partial a}\Delta a + \frac{\partial r}{\partial e}\Delta e + \frac{\partial r}{\partial M}\Delta M$$

$$= \Delta a \sum_{s=0}^{\infty} H_s \cos sM + a\Delta e \sum_{s=0}^{\infty} H'_s \cos sM - a\Delta M \sum_{s=0}^{\infty} sH_s \sin sM$$

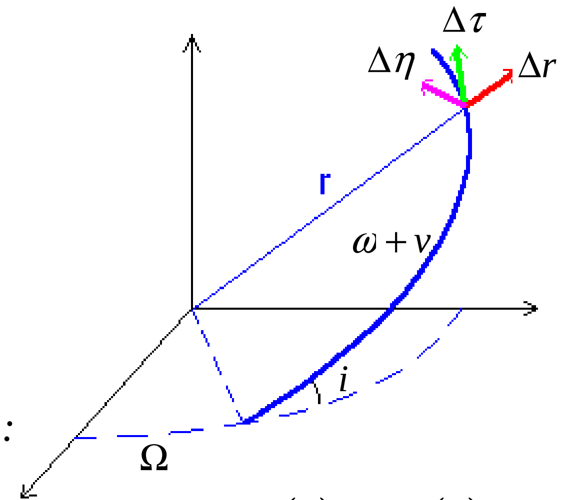


Transverse perturbation : $\Delta\tau = r[\Delta(\omega + v) + \Delta\Omega\cos i]$

$$\Delta\tau = r\left[\Delta\omega + \Delta M + \Delta\Omega\cos i + \Delta M \sum_{u=1}^{\infty} uI_u \cos uM + \Delta e \sum_{u=1}^{\infty} I'_u \sin uM\right]$$

Normal perturbation : $\Delta\eta = r[\Delta i \sin(\omega + v) - \Delta\Omega \sin i \cos(\omega + v)]$

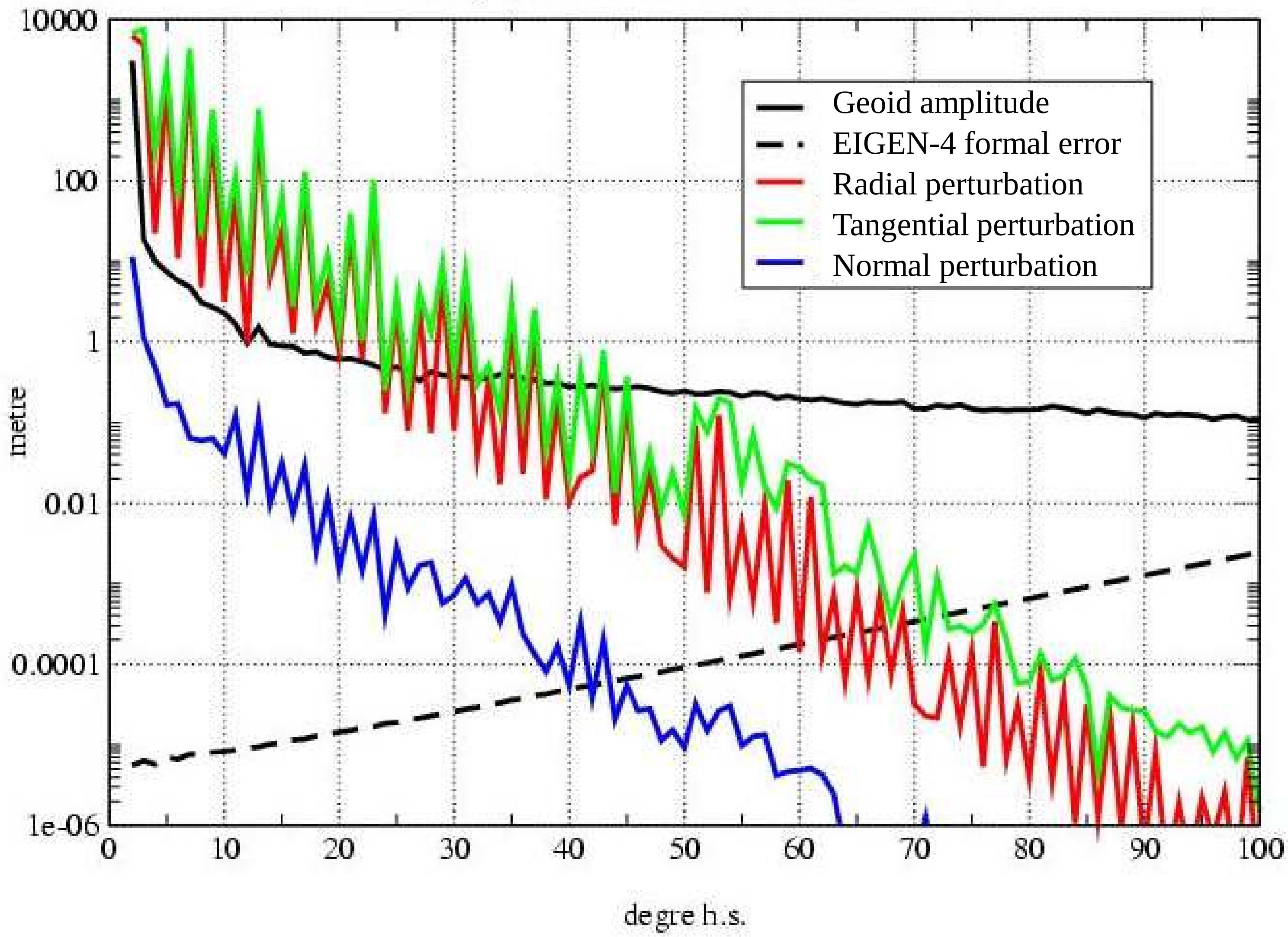
$$\Delta\eta = a\sum_{s=0}^{\infty} H_s \cos sM \left\{ \begin{array}{l} [\Delta i \sin\omega - \Delta\Omega \sin i \cos\omega] \sum_{u=0}^{\infty} R_u \cos uM \\ +[\Delta i \cos\omega + \Delta\Omega \sin i \sin\omega] \sum_{u=0}^{\infty} Q_u \sin uM \end{array} \right\}$$



where $H_s, H'_s = \frac{\partial H_s}{\partial e}, I_u, I'_u = \frac{\partial I_u}{\partial e}, R_u, Q_u$ *depend on Bessel's functions :*

$$J_s(x) = \frac{1}{2\pi}\int_0^{2\pi} \cos(st - x\sin t)\,dt \qquad \text{whose derivation property is :} \quad J'_s(x) = \frac{J_{s-1}(x) - J_{s+1}(x)}{2}$$

Spectral amplitude of geoid undulations and Jason orbit perturbations



Numerical vs. analytical methods

Numerical

• Numerical integration (of second order) of the fundamental equation of dynamics in Cartesian coordinates:

$$r = \iint \ddot{\bar{r}}\,dt \quad ; \quad \ddot{\bar{r}} = \sum_n \bar{A}(\bar{r}, \dot{\bar{r}}, \alpha_i)$$

from a set of initial conditions at t_0 (orbit and acceleration model parameters):

$$\bar{r}_0, \dot{\bar{r}}_0, \alpha_i = (C_{lm}, S_{lm}, \ldots, p_{dyn.})$$

• Adjustment of initial orbit parameters as well as of model parameters according to tracking observations:

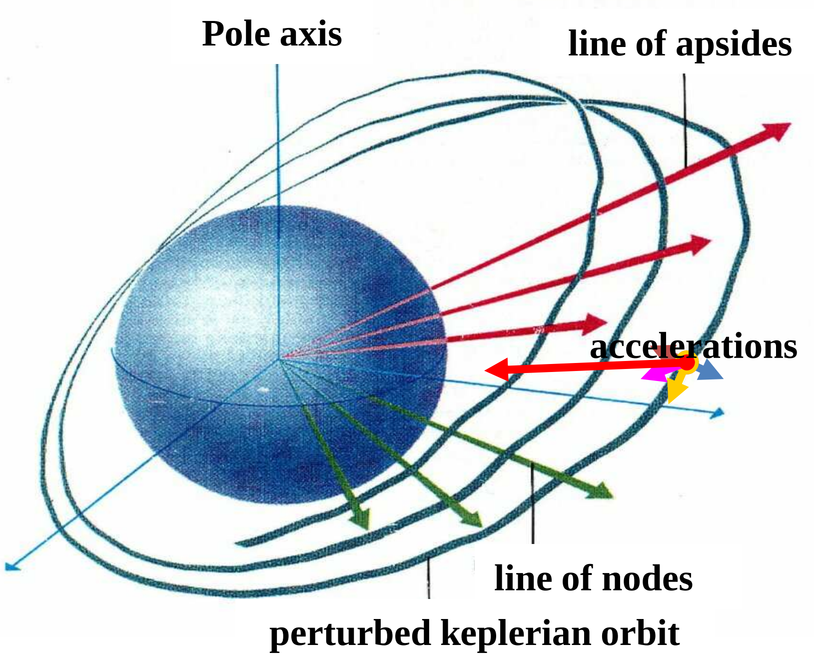
$$\Delta Q = Q_{obs} - Q_{calc} = \frac{\partial Q}{\partial \bar{r}_0}\Delta \bar{r}_0 + \frac{\partial Q}{\partial \dot{\bar{r}}_0}\Delta \dot{\bar{r}}_0 + \sum_i \frac{\partial Q}{\partial \alpha_i}\Delta \alpha_i$$

• Iterative method

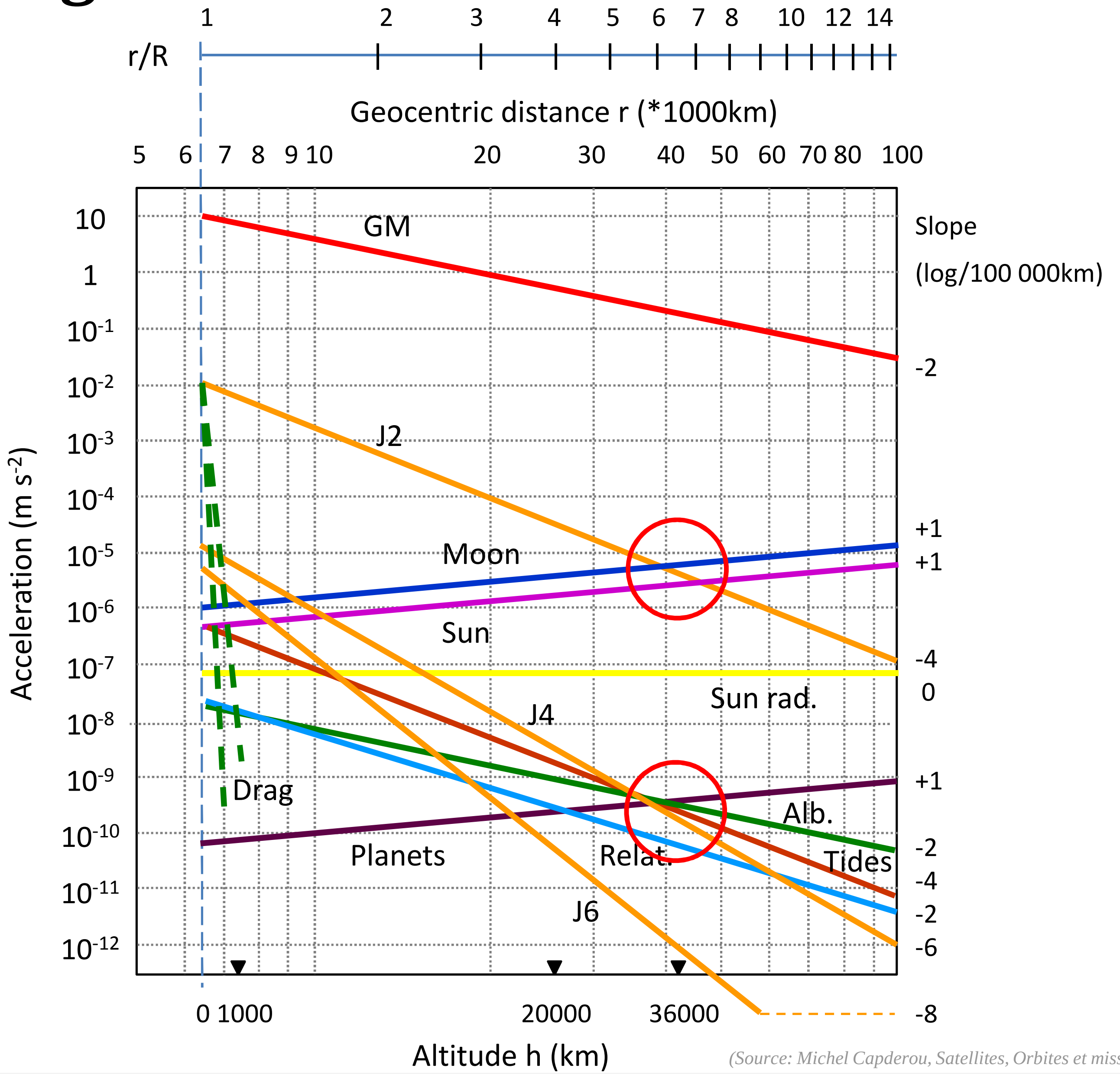
Analytical

• Integration (of first order) of Gauss / Lagrange’s equations in Keplerian elements

$$\left(\frac{da}{dt}, \frac{de}{dt}, \frac{di}{dt}, \frac{d\Omega}{dt}, \frac{d\omega}{dt}, \frac{dM}{dt}\right)$$



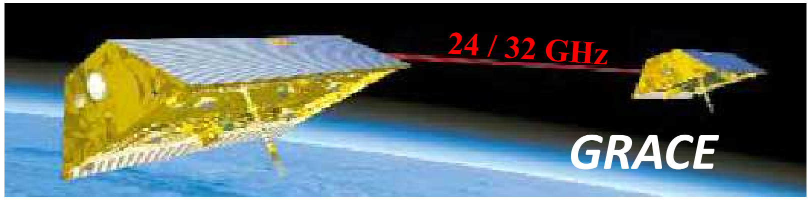
Perturbing accelerations

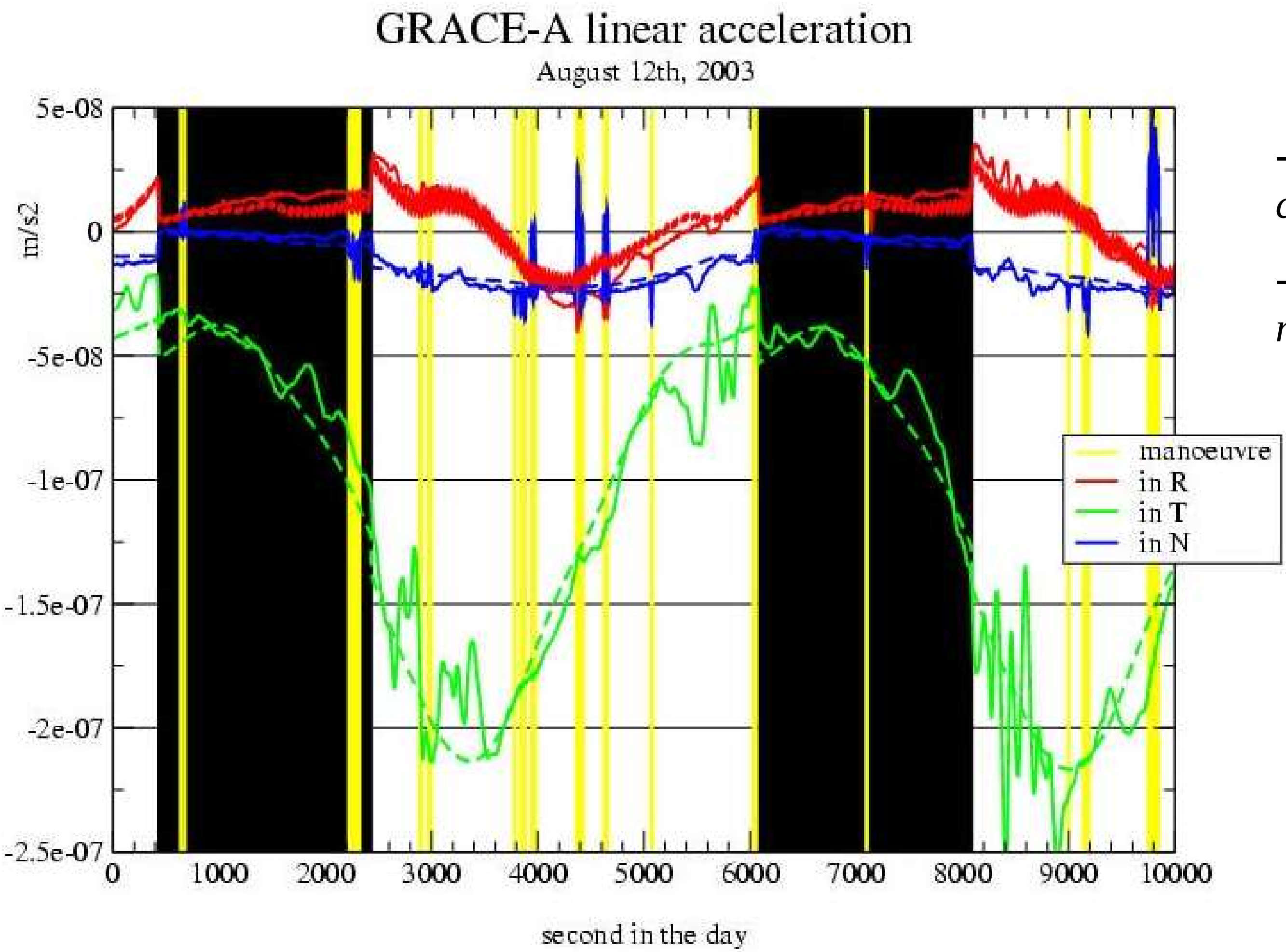


(Source: Michel Capderou, Satellites, Orbites et missions, Springer 2003)

Accelerometer

To measure the non gravitational accelerations



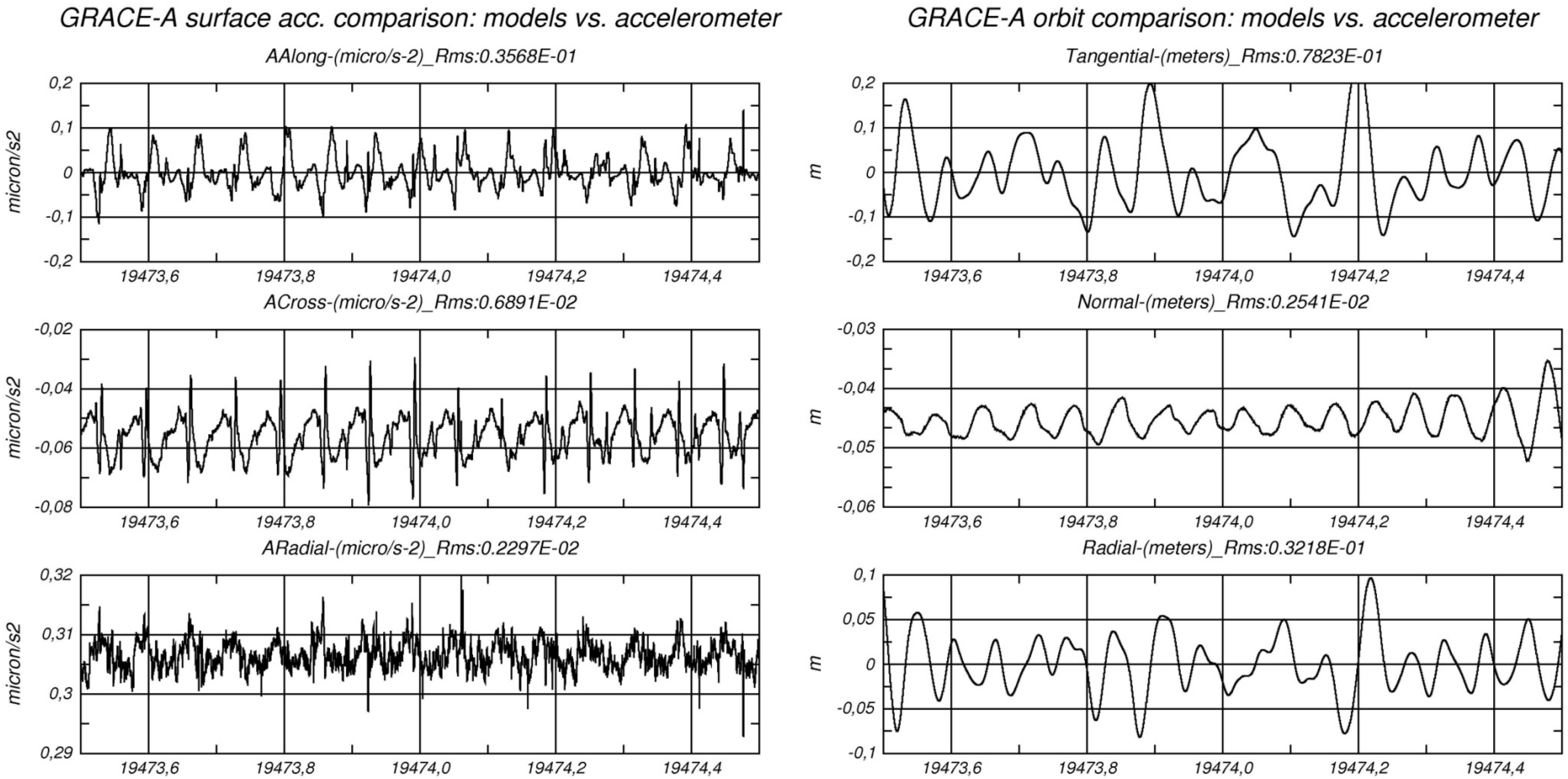


- *Accelerometer calibrated data: full line*
- *Non gravitational models: dashed line*

Empirical accelerations: additional accelerations to compensate modelling errors, mainly from non gravitational accelerations such as drag

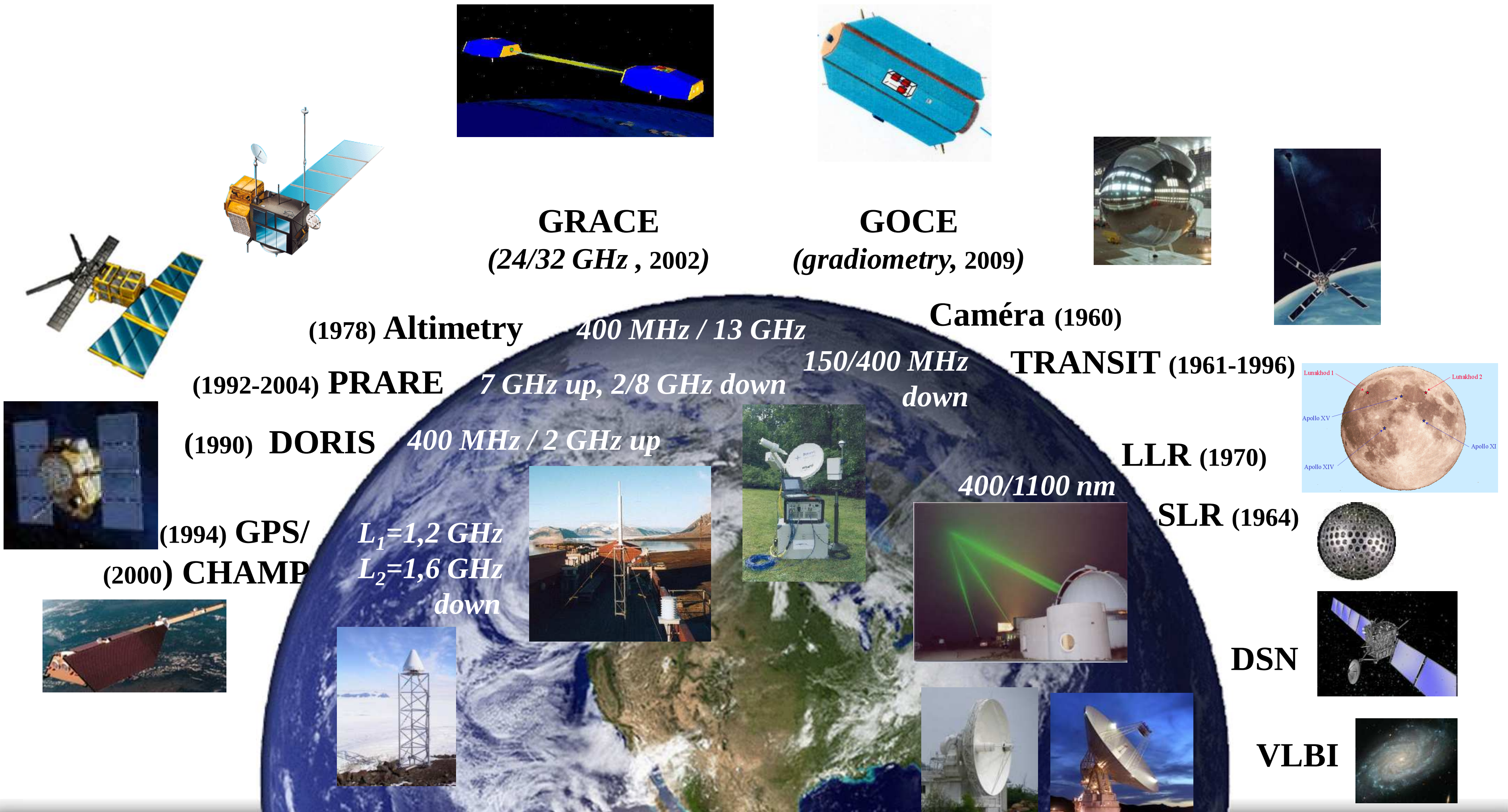
$$\underbrace{\boxed{\left(B+F.\Gamma_X^{STAR}\right)}}_{\text{Calibrated STAR acceleration / X}} \neq \underbrace{\boxed{\left(A_X^{drag}+A_X^{sun}+A_X^{alb}+A_X^{ir}\right)}}_{\text{Sum of non-gravitational accelerations / X}} \Rightarrow$$

use of empirical accelerations at the once per revolution period

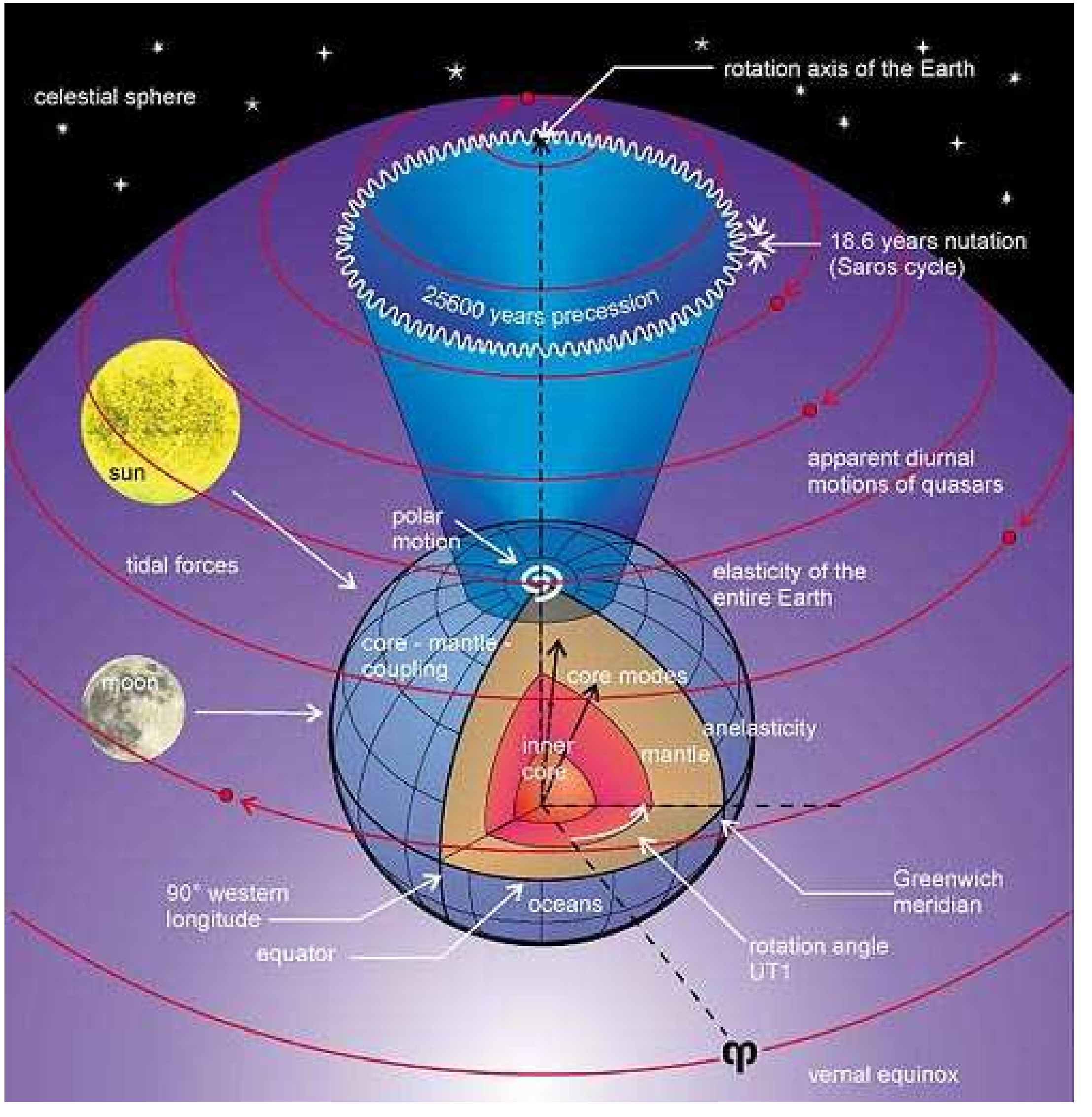


Space technique measurements

used in geodesy



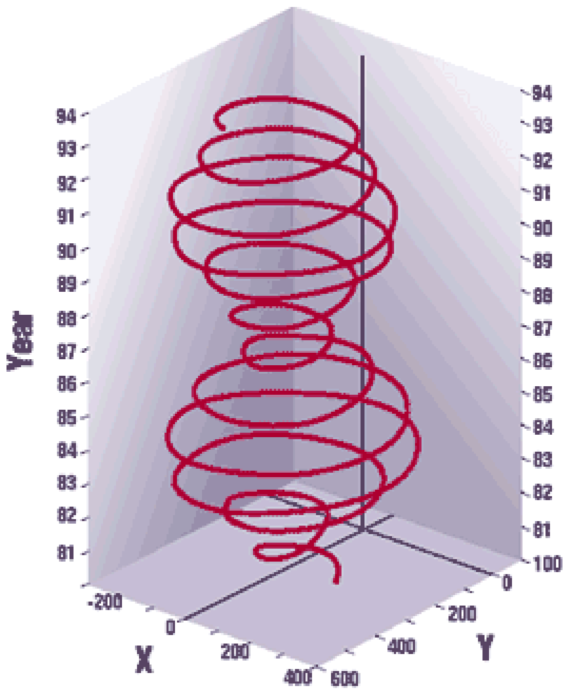
The Earth in space: precession, nutation

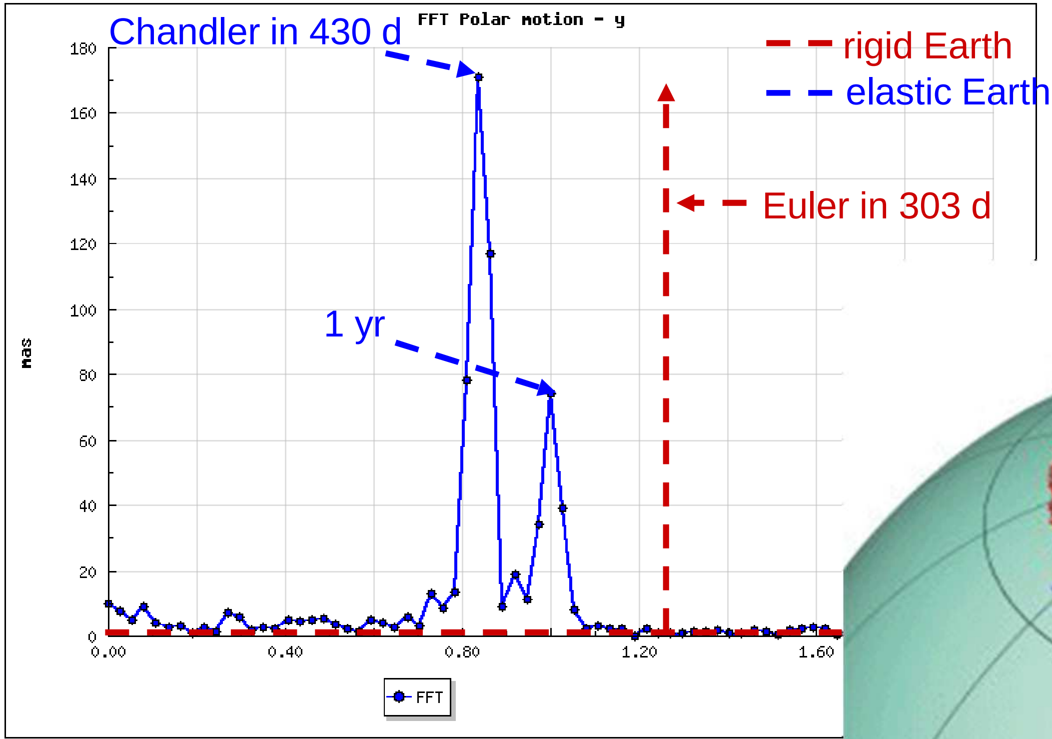


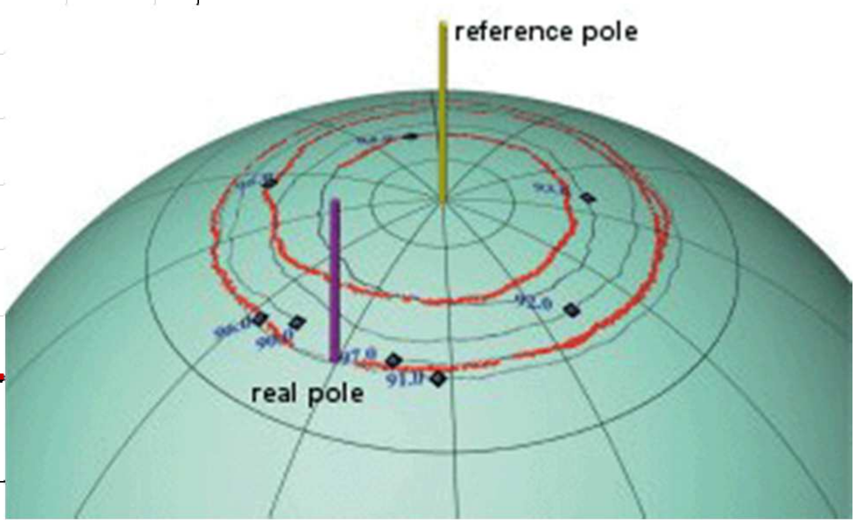
Polar motion

The polar motion has 3 principal components:

- the Chandler's oscillation at a 14 months period (~7 m)
- the annual oscillation caused by seasonal variations of the atmosphere and oceans (~3 m)
- the polar wander toward Canada (~10 cm/yr)





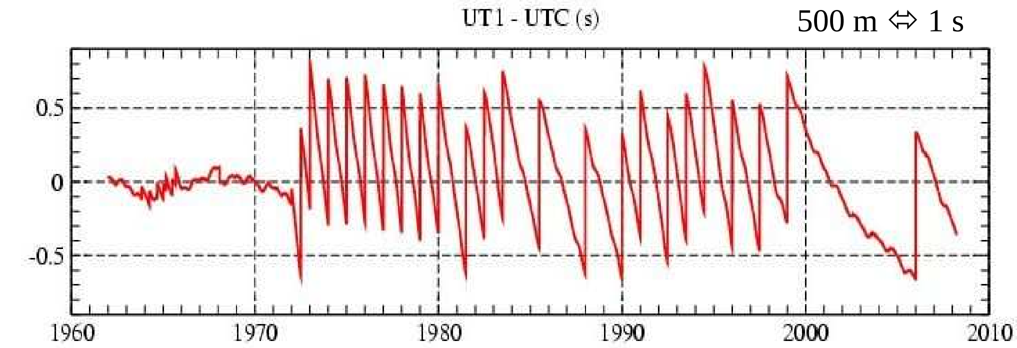


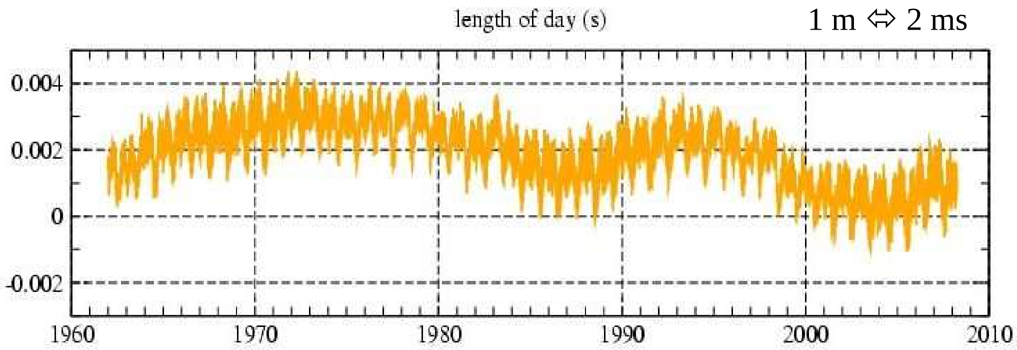
Earth’s rotation

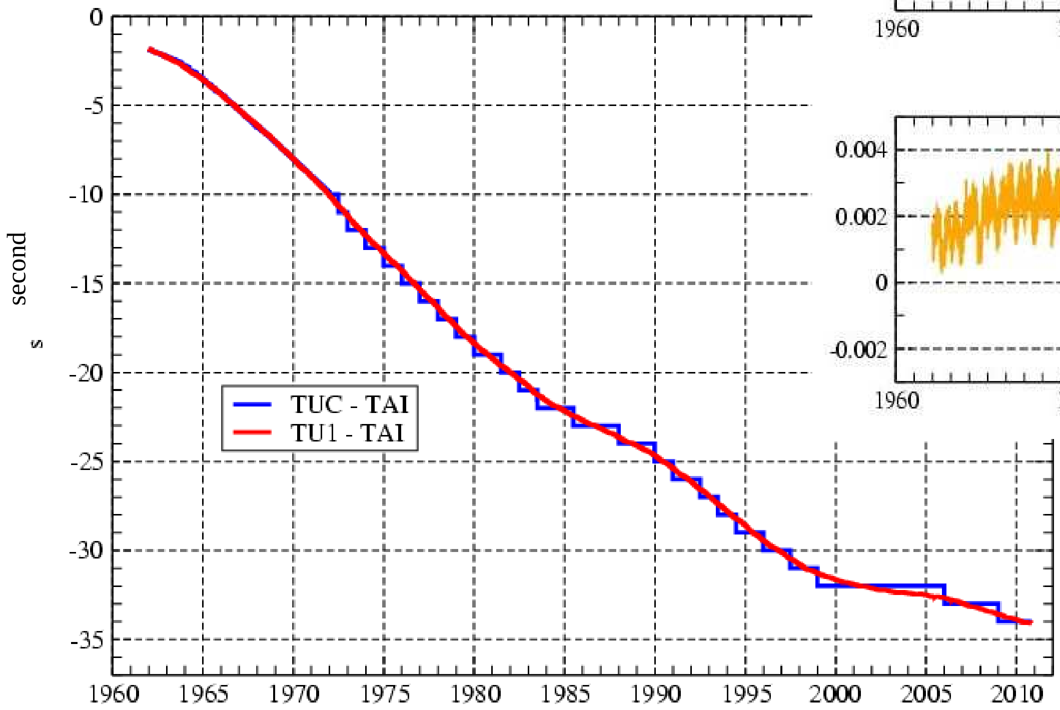
-0.9 s < UT1 – UTC < +0.9s

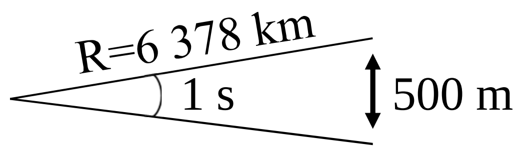
UTC = TAI + n

(TAI = UTC + 35s at 01/07/12)









2 billions years ago: 1 day = 10 hrs

400 millions years ago: 1 year = 400 days of 21 hrs

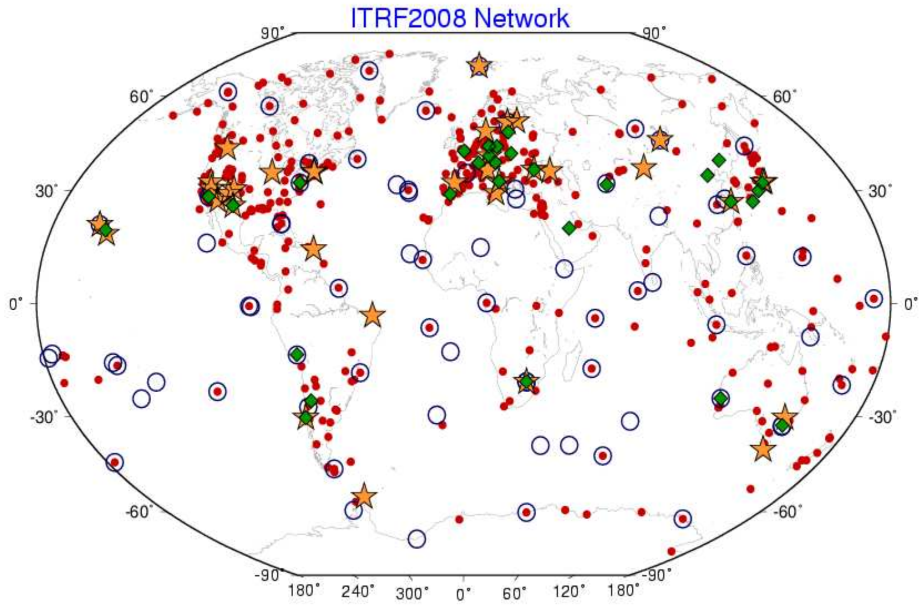
53 millions years ago: 1 year = 370 days of 24 hrs

Terrestrial reference systems

ITRF solutions (ITRF88, ITRF92, ITRF93, ITRF94, ITRF96, ITRF97, ITRF2000, ITRF2005, ITRF2008… ITRF2013) consist in sets of station positions and velocities with their variance/covariance matrices.
In the ITRF2008 release, Earth Orientation Parameters (EOPs) have been combined simultaneously with the station coordinates.

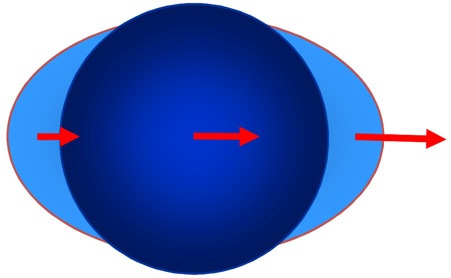
Input of ITRF2008:

Technique	Nb of sites	Time span
VLBI	84	1980.0 - 2009.0
SLR	89	1984.0 - 2009.0
GPS	492	1997.0 - 2009.5
DORIS	67	1993.0 - 2009.0

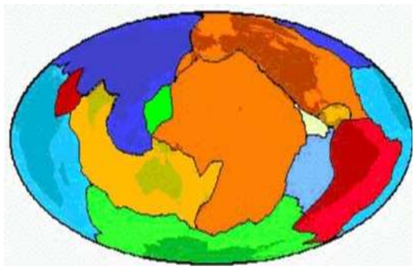


579 sites (920 stations)

Main Earth’s physical deformations



- tides (solid, oceanic…) → 30 cm
- surface loading → 10 cm
  - ocean tides
  - ocean currents
  - atmospheric pressure
  - hydrology
  - post-glacial rebound
- polar tide → 2 cm
- tectonics → 10 cm
- earthquakes → > m



Adjustment principle

Models : $\ddot{\bar{r}} = \sum_n \bar{A}(\bar{r}, \dot{\bar{r}}, \alpha_i)$

Initialization : $\bar{r}_0, \dot{\bar{r}}_0, \alpha_i = (C_{lm}, S_{lm}, \ldots, p_{dyn.})$

Integration :

$$\frac{d^2\bar{r}}{dt^2} = \sum_n \bar{A}(\bar{r}, \dot{\bar{r}}, \alpha_i)$$

$$\frac{d^2}{dt^2}\left(\frac{\partial\bar{r}}{\partial\bar{r}_0}\right) = \frac{\partial\ddot{\bar{r}}}{\partial\bar{r}_0} = \sum_n \frac{\partial\bar{A}}{\partial\bar{r}}\frac{\partial\bar{r}}{\partial\bar{r}_0} + \sum_n \frac{\partial\bar{A}}{\partial\dot{\bar{r}}}\frac{d}{dt}\left(\frac{\partial\bar{r}}{\partial\bar{r}_0}\right)$$

$$\frac{d^2}{dt^2}\left(\frac{\partial\bar{r}}{\partial\alpha_i}\right) = \frac{\partial\ddot{\bar{r}}}{\partial\alpha_i} = \sum_n \frac{\partial\bar{A}}{\partial\bar{r}}\frac{\partial\bar{r}}{\partial\alpha_i} + \sum_n \frac{\partial\bar{A}}{\partial\dot{\bar{r}}}\frac{d}{dt}\left(\frac{\partial\bar{r}}{\partial\alpha_i}\right) + \sum_n \frac{\partial\bar{A}}{\partial\alpha_i}$$

Observations : $Q_{obs} = Q_{calc}(\bar{r}, \beta_j) + residual$

$$\beta_j = (\bar{R}_{sta}, \ldots, x_p, y_p, UT1, \ldots, f_r, \ldots)$$

$$\frac{\partial Q}{\partial\bar{r}_0} = \frac{\partial Q}{\partial\bar{r}}\frac{\partial\bar{r}}{\partial\bar{r}_0}, \quad \frac{\partial Q}{\partial\dot{\bar{r}}_0} = \frac{\partial Q}{\partial\bar{r}}\frac{\partial\bar{r}}{\partial\dot{\bar{r}}_0}, \quad \frac{\partial Q}{\partial\alpha_i} = \frac{\partial Q}{\partial\bar{r}}\frac{\partial\bar{r}}{\partial\alpha_i}, \quad \frac{\partial Q}{\partial\beta_j}$$

Weighting : $Q' = Q/\sigma$

Adequacy : $\Delta Q' = Q'_{obs} - Q'_{calc} = \frac{\partial Q'}{\partial\bar{r}_0}\Delta\bar{r}_0 + \frac{\partial Q'}{\partial\dot{\bar{r}}_0}\Delta\dot{\bar{r}}_0 + \sum_i \frac{\partial Q'}{\partial\alpha_i}\Delta\alpha_i + \sum_j \frac{\partial Q'}{\partial\beta_j}\Delta\beta_j$

Resolution : $\bar{r}_0 + \Delta\bar{r}_0, \quad \dot{\bar{r}}_0 + \Delta\dot{\bar{r}}_0, \quad \alpha_i + \Delta\alpha_i, \quad \beta_j + \Delta\beta_j$

Iteration : $convergence\ test : \frac{\sum_i \Delta Q_k'^2 - \sum_i \Delta Q_{k-1}'^2}{\sum_i \Delta Q_{k-1}'^2} < \varepsilon$

Geodetic parameters (*to be adjusted in GINS)*

GÉODÉSIE PAR INTÉGRATIONS NUMÉRIQUES SIMULTANÉES

Acceleration parameters

- initial orbit elements $(x, y, z, \dot{x}, \dot{y}, \dot{z})$ *or* $(a, e, i, \omega, \Omega, M)$
- Stokes coefficients (C_{nm}, S_{nm})
- Love numbers $(h_n, l_n, k_n \ldots)$
- ocean tide coefficients
- drag, radiation pressure scale factors
- drag model parameters
- empirical coefficients (bias, scale, periodic terms)

Geometrical parameters

- station, geocenter, quasar coordinates
- pole coordinates, UT1, nutation corrections
- mean sea surface coefficients

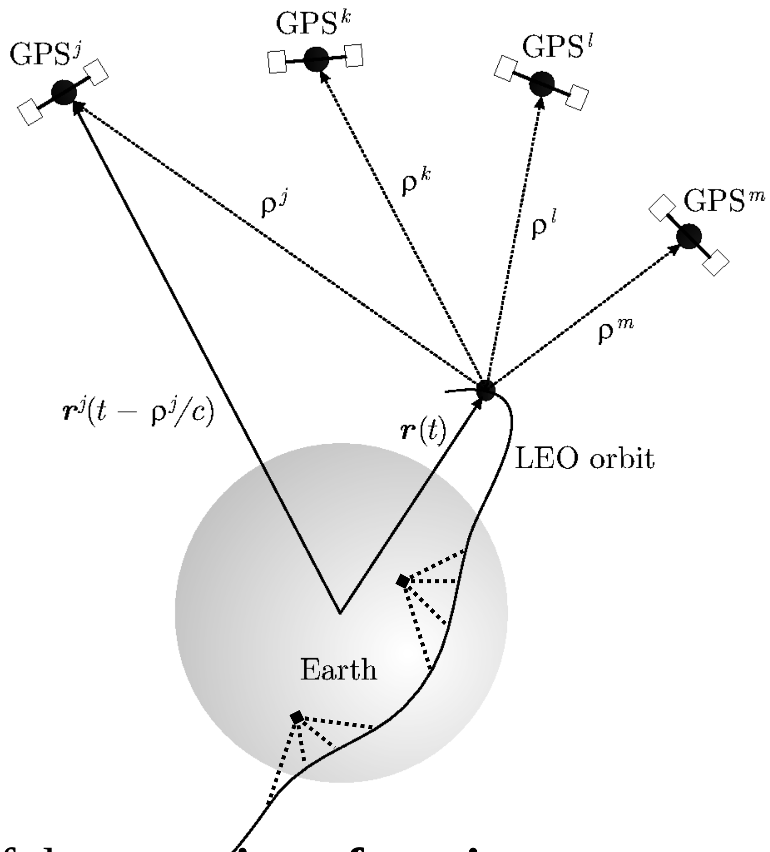
Measurement parameters

- range/time biases
- tropospheric zenithal biases and gradients (GNSS)
- GNSS clock and ambiguity parameters

Dynamic orbit representation

The **acceleration** consists of **gravitational and non-gravitational perturbations** taken into account to model the satellite trajectory:

$$\ddot{\bar{r}} = -GM\,\frac{\bar{r}}{r^3} + \bar{\gamma}(t, \bar{r}, \dot{\bar{r}}, \alpha_i, \beta_j)$$



Unknown parameters α_i of force models, or β_j of geometrical models may appear in the equation of motion together with deterministic (known) accelerations given by analytical models.

Satellite trajectory is a particular solution of the **equation of motion**

- one set of **initial conditions** (orbital elements) is estimated per arc
- dynamical parameters of force models on request
- geometrical parameters of Earth or measurements (e.g. pole coordinates, stations coordinates, clock biases, troposphere delays…)

Deterministic parameters

Deterministic accelerations express mainly drag or radiation pressure scale factors, constant or piecewise linear biases or even once/twice per revolution forces :

$$\overline{A}(t,\alpha) = \overline{A}(t_j)\frac{t_{j+1} - t}{t_{j+1} - t_j} + \overline{A}(t_{j+1})\frac{t - t_j}{t_{j+1} - t_j} \qquad t \in \left[t_j, t_{j+1}\right]$$

$$\overline{A}(t) = \overline{B} + \overline{A}_C \cos(\omega t) + \overline{A}_S \sin(\omega t) \qquad \omega = \frac{2\pi}{T_{rev}}$$

Derivatives with respect to parameters α_i (=$A(t_i)$, B, A_C, A_S…) are numerically integrated in variational equations together with the motion equation :

$$\frac{d^2}{dt^2}\left(\frac{\partial \overline{r}}{\partial \alpha_i}\right) = \frac{\partial \ddot{\overline{r}}}{\partial \alpha_i} = \sum_n \frac{\partial \overline{A}}{\partial \overline{r}}\frac{\partial \overline{r}}{\partial \alpha_i} + \sum_n \frac{\partial \overline{A}}{\partial \dot{\overline{r}}}\frac{d}{dt}\left(\frac{\partial \overline{r}}{\partial \alpha_i}\right) + \sum_n \frac{\partial \overline{A}}{\partial \alpha_i}$$

Then combined with the observation equation expanded in Taylor’s series at first order :

$$\Delta Q = Q_{obs} - Q_{calc}(\overline{r}, \alpha_i) = \frac{\partial Q}{\partial \overline{r}_0}\Delta \overline{r}_0 + \frac{\partial Q}{\partial \dot{\overline{r}}_0}\Delta \dot{\overline{r}}_0 + \sum_i \frac{\partial Q}{\partial \alpha_i}\Delta \alpha_i$$

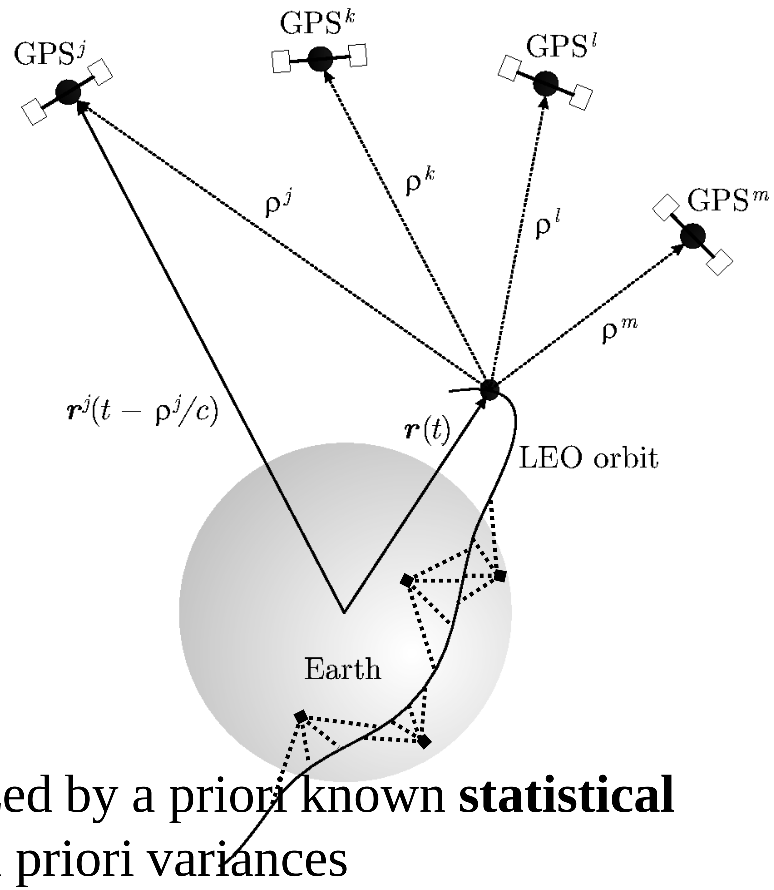
$$\text{with} \quad \frac{\partial Q}{\partial \alpha_i} = \frac{\partial Q_{calc}}{\partial \overline{r}}\frac{\partial \overline{r}}{\partial \alpha_i}$$

Reduced-Dynamic orbit representation

Additional **empirical accelerations** are taken into account to model the satellite trajectory :

$$\ddot{\bar{r}} = -GM\,\frac{\bar{r}}{r^3} + \bar{\gamma}(t,\bar{r},\dot{\bar{r}},\alpha_i,\beta_j,\delta_k)$$

e.g. piece-wise constant accelerations (every 6 min) in RTN, constrained with σ=2.0*10^{-8} m/s^2 for the GOCE orbit.



Pseudo-stochastic parameters *δ_k* are :

- additional empirical parameters characterized by a priori known **statistical properties**, e.g., by expectation values and a priori variances
- useful to compensate for deficiencies in dynamic models, e.g., deficiencies in models describing non-gravitational accelerations
- often set up as **piecewise constant accelerations** to ensure that satellite trajectories are continuous and differentiable at any epoch

Stochastic parameters

Stochastic accelerations F are expressed in terms of a 1st order Gauss-Markov process. They are correlated trough the relationship :

$$F(t_{j+1}) = e^{-\frac{t_{j+1}-t_j}{\tau}} F(t_j) + \eta_j \qquad with \qquad cov(\eta_j) = e^{-2\frac{t_{j+1}-t_j}{\tau}} \sigma^2$$

τ represents the correlation distance (e.g. $\tau=3\Delta t$) and η_j the uncertainties of the accelerations (e.g. $\sigma=10^{-8}\ ms^{-2}$).

The constraint equation : $AF = 0 \quad with \quad F = [F(t_0), F(t_1)..., F(t_{n-1}), F(t_n)]^T$

transforms by recurrence into normal matrix : $A^T A F = 0$

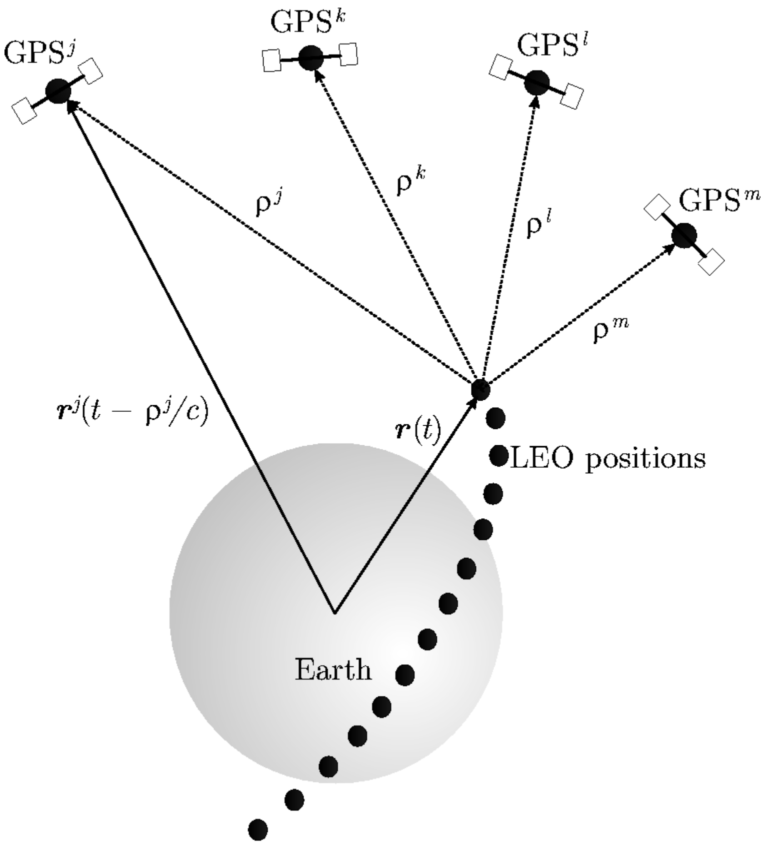
$$to\ be\ multiplied\ by: \quad \sigma^{-2}(1-e^{-2\frac{t_{j+1}-t_j}{\tau}})^{-1}$$

$$and \quad A = \begin{bmatrix} e^{-\frac{t_1-t_0}{\tau}} & -1 & 0 & \cdots & 0 \\ 0 & e^{-\frac{t_2-t_1}{\tau}} & -1 & & \vdots \\ & & \ddots & & \\ \vdots & & e^{-\frac{t_{n-1}-t_{n-2}}{\tau}} & -1 & 0 \\ 0 \ \cdots \ 0 & & e^{-\frac{t_n-t_{n-1}}{\tau}} & & -1 \end{bmatrix} \Rightarrow A^T A = \begin{bmatrix} e^{-2\frac{t_1-t_0}{\tau}} & -e^{-\frac{t_1-t_0}{\tau}} & 0 & \cdots & 0 \\ -e^{-\frac{t_1-t_0}{\tau}} & 1+e^{-2\frac{t_2-t_1}{\tau}} & -e^{-\frac{t_2-t_1}{\tau}} & & \vdots \\ & & \ddots & & \\ \vdots & & -e^{-\frac{t_{n-1}-t_{n-2}}{\tau}} & 1+e^{-2\frac{t_n-t_{n-1}}{\tau}} & -e^{-\frac{t_n-t_{n-1}}{\tau}} \\ 0 & \cdots & 0 & -e^{-\frac{t_n-t_{n-1}}{\tau}} & -1 \end{bmatrix}$$

Kinematic orbit representation

A kinematic orbit is an ephemeris at discrete measurement epochs **fully independent on the force models** used for LEO orbit determination.

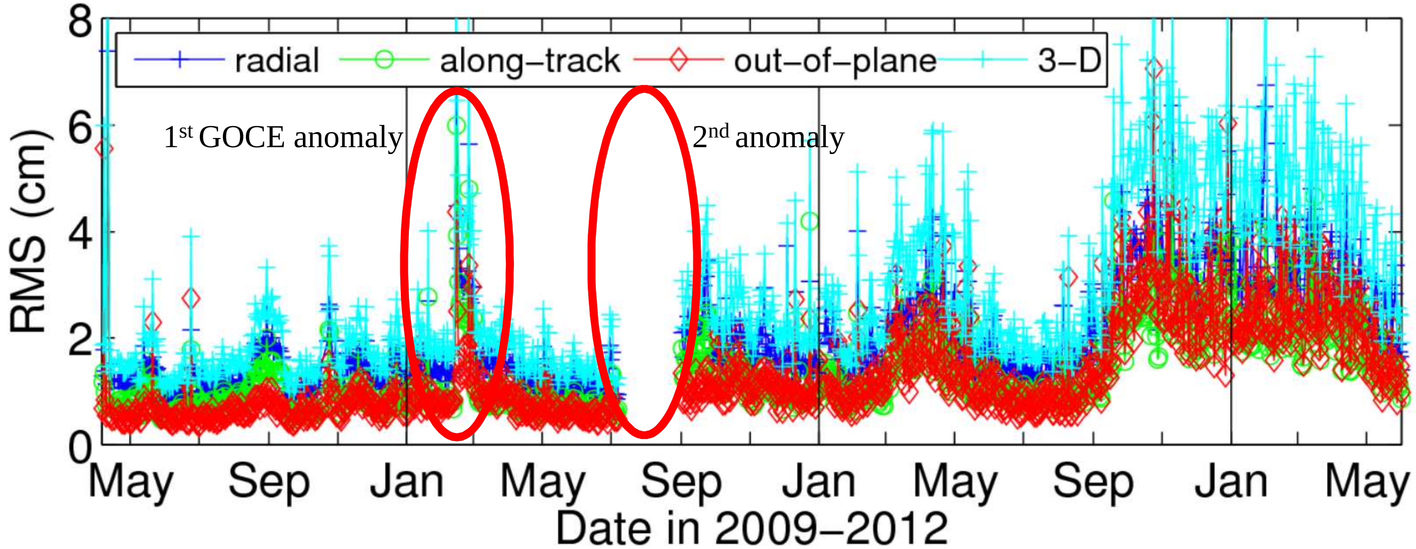
The kinematic approach is mainly appropriate with GNSS measurements (with overabundant observation data), although it is **very sensitive to measurement noise and data gaps**.

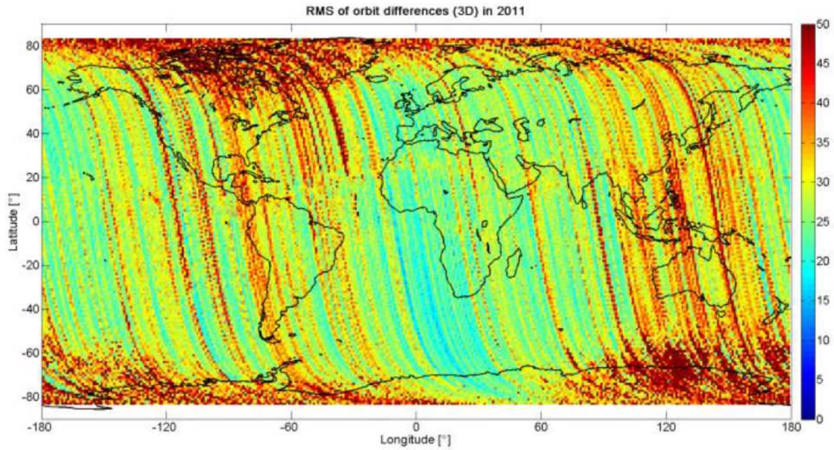


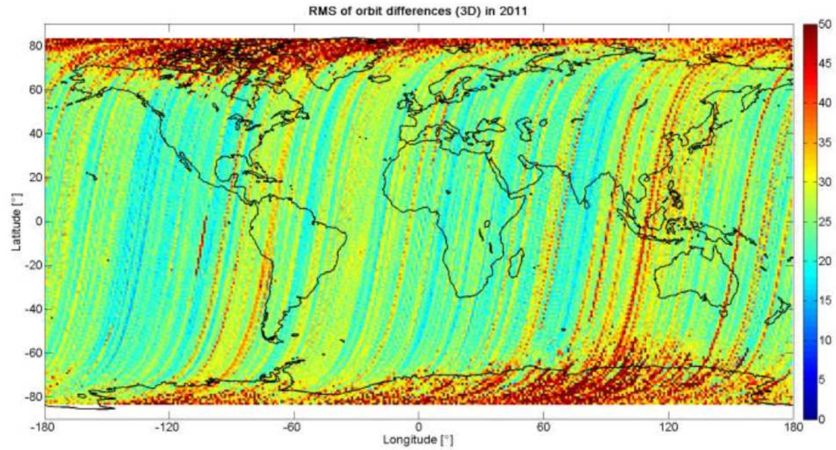
Kinematic positions are estimated **for each measurement epoch**:
- measurement epochs do not need to be identical with nominal epochs
- positions are **independent of models** describing the LEO dynamics
- velocities cannot be provided in a strict sense

Kin. / Red. Dyn. orbit differences

GOCE orbit comparison with the Bernese software





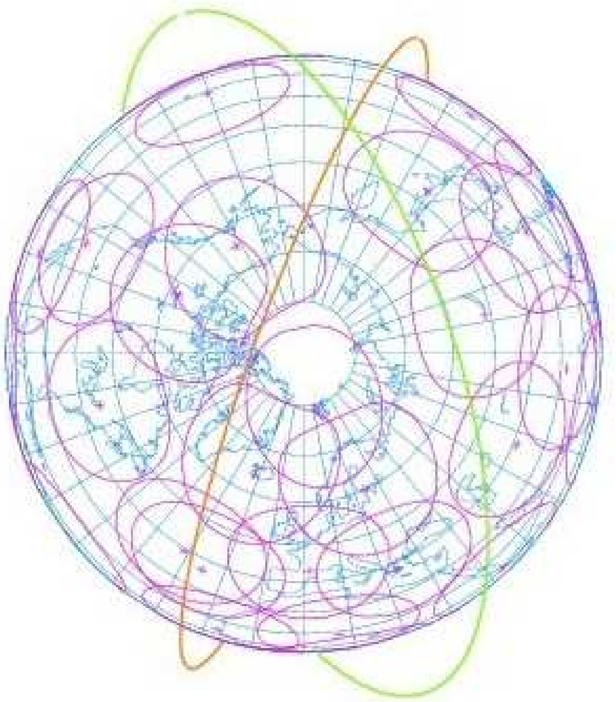


A. Jäggi, AIUB, 2012

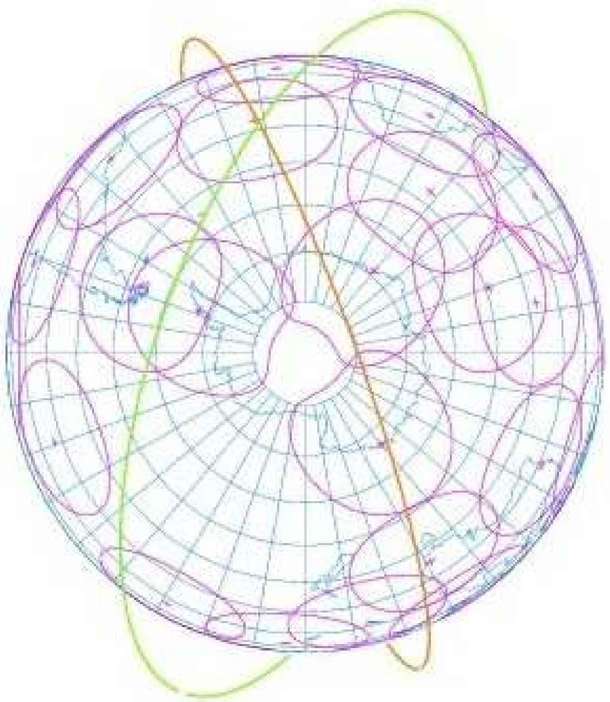
Orbit classes

• a:	< 1500 km	LEO	(Low Earth Orbit)
	~ 20000km	MEO	(Medium Earth Orbit)
	~ 36000km	GEO	(Geostationary Earth Orbit)
• e:	~ 0	circular or quasi circular	
	>> 0 (and <1)	eccentric (HEO: Highly Eccentric Orbit, GTO: Geostationary Transfer orbit)	
• i:	< 90 deg	prograde (in the direction of the Earth rotation)	
	~ 90 deg	polar or quasi polar	
	> 90 deg	retrograde (opposite to the Earth rotation)	

DORIS network visibility circles



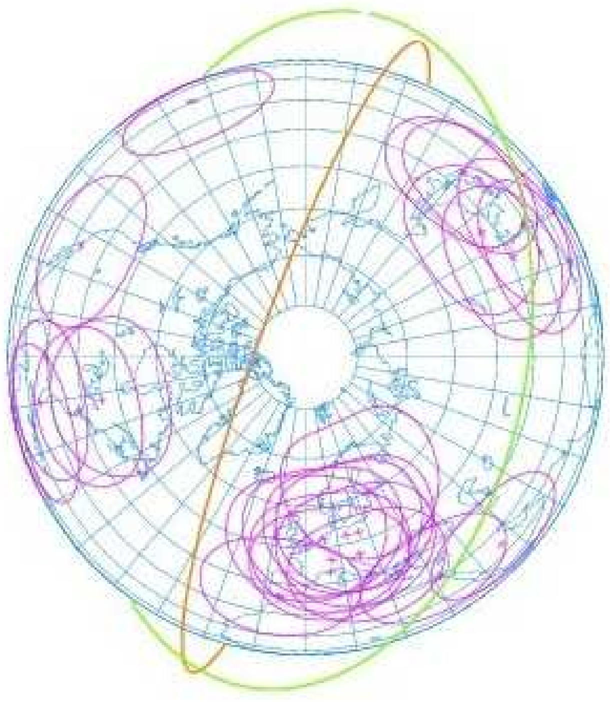
DORIS north hemisphere



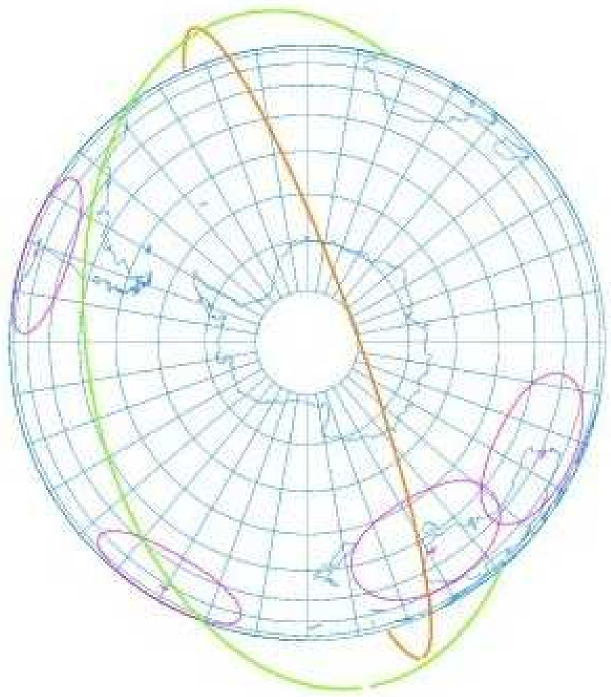
DORIS south hemisphere

SPOT (98.5 deg) and Jason (66 deg)

SLR network visibility circles



LASER north hemisphere



LASER south hemisphere

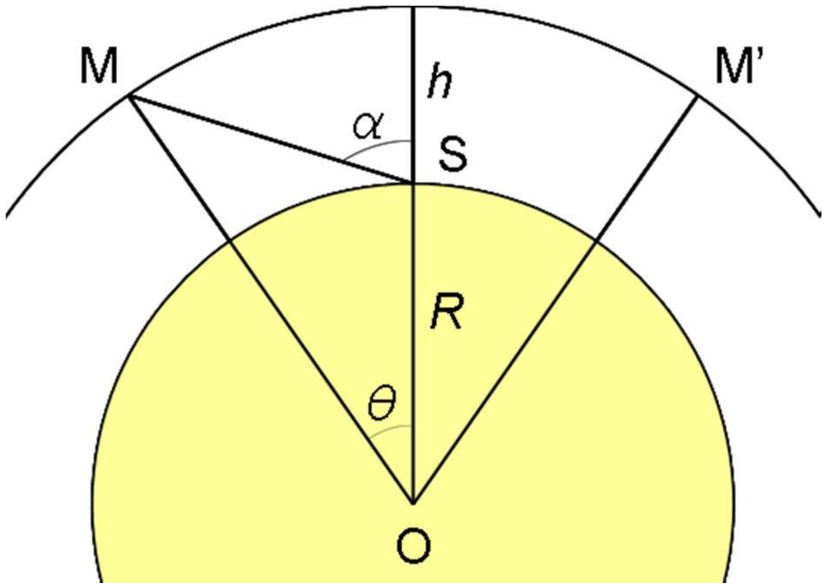
Starlette (50 deg) and Stella (98 deg)

Pass duration

In the OSM triangle:

$$\frac{\sin\alpha}{R+h}=\frac{\sin(\alpha-\theta)}{R}\Rightarrow \cos\theta-\frac{\sin\theta}{\tan\alpha}=\frac{R}{R+h}$$

$$\Delta t = 2\theta\sqrt{\frac{a^3}{GM}}$$



Transformation after squaring:

$$\tan\theta=\frac{\sin\alpha\left(\sqrt{\left(\frac{R+h}{R}\right)^2-\sin^2\alpha}-\cos\alpha\right)}{1+\cos\alpha\left(\sqrt{\left(\frac{R+h}{R}\right)^2-\sin^2\alpha}-\cos\alpha\right)}$$

Maximum pass duration (3rd Kepler law):

$$n=\frac{2\pi}{T}=\sqrt{\frac{GM}{a^3}} \quad\Rightarrow\quad \Delta t = 2\theta\sqrt{\frac{a^3}{GM}}$$

Curvilinear diameter of the visibility circle (on the track) :

$$D = 2\theta\, R$$

α : *zenital angle*

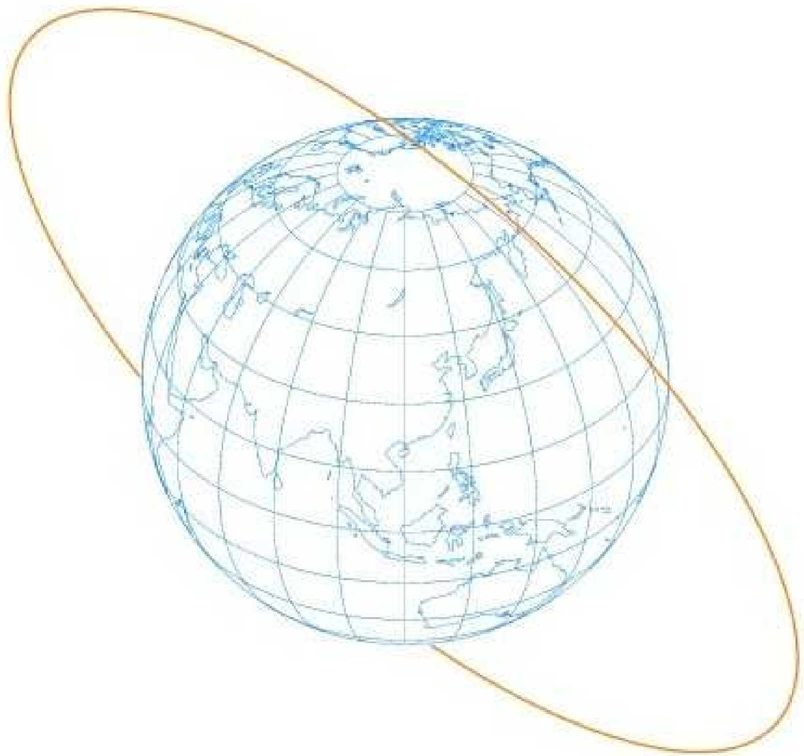
θ : *orbital angle*

Pass duration

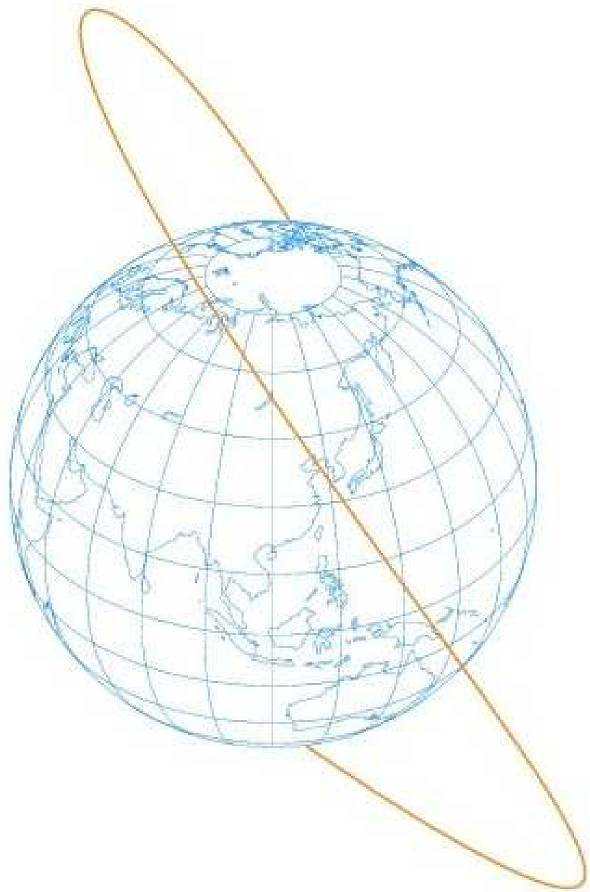
for an elevation angle cutoff of 10 deg. corresponding to α = 80 deg.

Satellite	*h (km)*	*2θ (deg.)*	*Δt*	*D (km)*
GOCE	260	17	4.5 mn	1900
GRACE	460	26	7 mn	3000
SPOT	830	39	11 mn	4300
Jason	1330	50	15 mn	5500
LAGEOS	5900	98	1 h	10700
GPS	20000	132	6.5 h	14700
Meteosat	35800	160	24 h	17700

Reference frames



Inertial reference frame



Earth fixed reference frame

Equatorial drift of the node

Difference between 2 consecutive ascending nodes:

In one orbit revolution (period ΔT_{orb}) the ascending node rotates $\dot{\Omega}\Delta T_{orb}$ and the Earth rotates eastward $\dot{\Omega}_E \Delta T_{orb}$, hence the node precesses westward by:

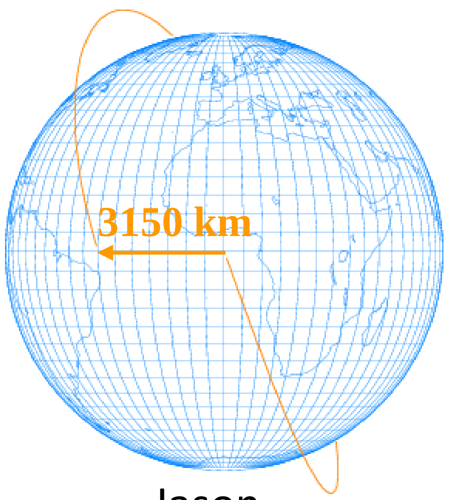
$$\boxed{\Delta\lambda = \left(\dot{\Omega} - \dot{\Omega}_E\right) T_{orb}}$$

$$\dot{\Omega}_E = 7.292115\,10^{-5}\ rad/s$$
$$= 360.9856\ deg/day$$

For a **sunsynchronous** satellite $\left(\dot{\Omega} = \dot{\Omega}_S\right)$:

$$\Delta\lambda = \left(\dot{\Omega}_S - \dot{\Omega}_E\right) T_{orb} = -\frac{2\pi}{86400} T_{orb}$$

$$(\dot{\Omega}_S = 0.9856\ deg/day)$$



	$d\Omega/dt$ (deg/day)	T_{orb} (min)	$R\Delta\lambda$ (km)
GOCE	0.985	89.67	2500
SPOT	0.986	101.46	2820
Jason	-2.079	112.43	3150
LAGEOS	0.342	225.57	6280
GPS	-0.047	711.46	19800

2.4.3.1. Equator Crossing Longitudes (in order of Pass Number)

Pass	Longitude	Pass	Longitude	Pass	Longitude	Pass	Longitude	Pass	Longitude	Pass	Longitude
1	99.9249	44	30.4744	87	321.0274	130	251.5783	173	182.1282	216	112.6813
2	265.7517	45	196.3012	88	126.8541	131	57.4042	174	347.9550	217	278.5075
3	71.5776	46	2.1280	89	292.6810	132	223.2310	175	153.7829	218	84.3343
4	237.4044	47	167.9557	90	98.5078	133	29.0576	176	319.6096	219	250.1600
5	43.2305	48	333.7825	91	264.3336	134	194.8843	177	125.4369	220	55.9867
6	209.0573	49	139.6102	92	70.1603	135	0.7117	178	291.2636	221	221.8129
7	14.8844	50	305.4370	93	235.9862	136	166.5385	179	97.0902	222	27.6397
8	180.7112	51	111.2637	94	41.8130	137	332.3659	180	262.9170	223	193.4666
9	346.5387	52	277.0905	95	207.6395	138	138.1927	181	68.7430	224	359.2934
10	152.3655	53	82.9167	96	13.4663	139	304.0198	182	234.5697	225	165.1212
11	318.1928	54	248.7435	97	179.2937	140	109.8466	183	40.3959	226	330.9479
12	124.0196	55	54.5694	98	345.1205	141	275.6727	184	206.2227	227	136.7755
13	289.8463	56	220.3962	99	150.9484	142	81.4995	185	12.0499	228	302.6023
14	95.6731	57	26.2229	100	316.7751	143	247.3252	186	177.8767	229	108.4290
15	261.4989	58	192.0497	101	122.6022	144	53.1520	187	343.7042	230	274.2558
16	67.3256	59	357.8771	102	288.4290	145	218.9782	188	149.5309	231	80.0819
17	233.1515	60	163.7039	103	94.2556	146	24.8050	189	315.3582	232	245.9087
18	38.9783	61	329.5313	104	260.0823	147	190.6320	190	121.1850	233	51.7347
19	204.8049	62	135.3580	105	65.9083	148	356.4588	191	287.0117	234	217.5614

Equator crossing

Apparent inclination:

In the tangent plane to the sphere Earth at the point representing the ascending node on a parallel, one can express the angular velocity such as:

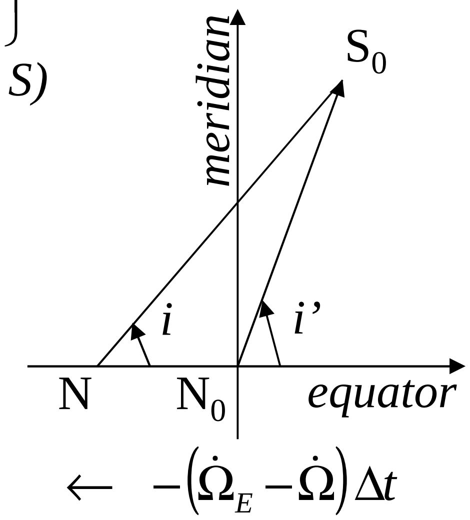
• along the orbit in the non-rotating frame: $NS_0 \left| \begin{matrix} n\cos i \\ n\sin i \end{matrix} \right.$

• along the equator toward the rotating frame: $N_0N \left| \begin{matrix} \dot{\Omega} - \dot{\Omega}_E \\ 0 \end{matrix} \right.$

$$\Rightarrow N_0S \left| \begin{matrix} n\cos i + \left(\dot{\Omega} - \dot{\Omega}_E\right) \\ n\sin i \end{matrix} \right.$$

(After an infinitesimal time Δt, N_0 becomes N and S_0 becomes S)

$$\Rightarrow \boxed{\tan i' = \frac{\sin i}{\cos i - \left(\dot{\Omega}_E - \dot{\Omega}\right)/n}} = \frac{\sin i}{\cos i - T_{orb}\left(\dot{\Omega}_E - \dot{\Omega}\right)/2\pi}$$

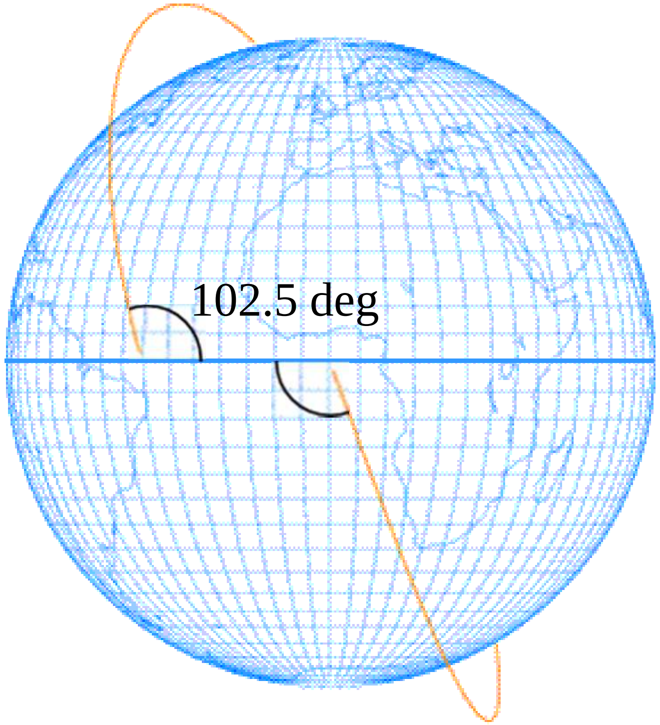


For a **sunsynchronous** satellite $\left(\dot{\Omega} = \dot{\Omega}_S, T_{orb} \text{ in } rev/day\right)$:

$$\tan i' = \frac{\sin i}{\cos i - T_{orb}} \quad \Rightarrow \quad i' - i \approx 4\,deg \quad (SPOT)$$

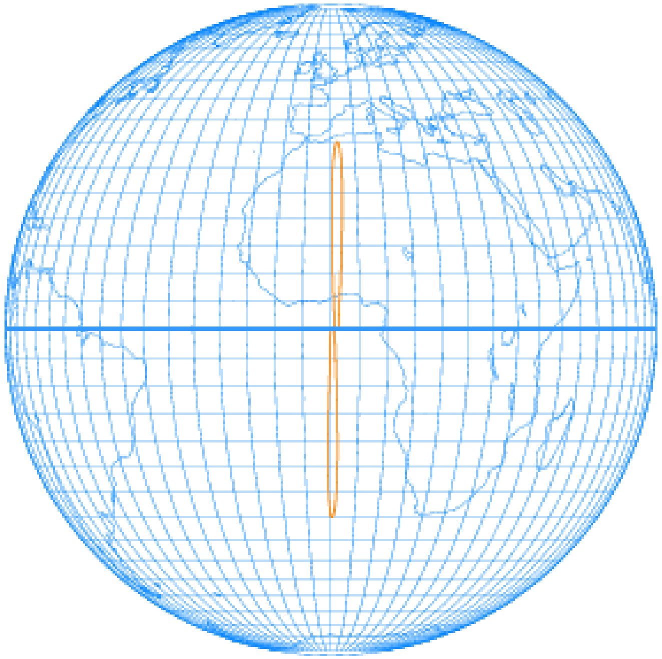
For a **geostationary** satellite $\left(n \approx \dot{\Omega}_E, \dot{\Omega} \approx 0, T_{orb} \approx 1\right)$:

$$\tan i' = \frac{\sin i}{\cos i - 1} = \tan\left(\frac{\pi}{2} + \frac{i}{2}\right) \quad \Rightarrow \quad i' \approx 90\,deg + \frac{i}{2} \quad (Lemniscate\ curve)$$



SPOT

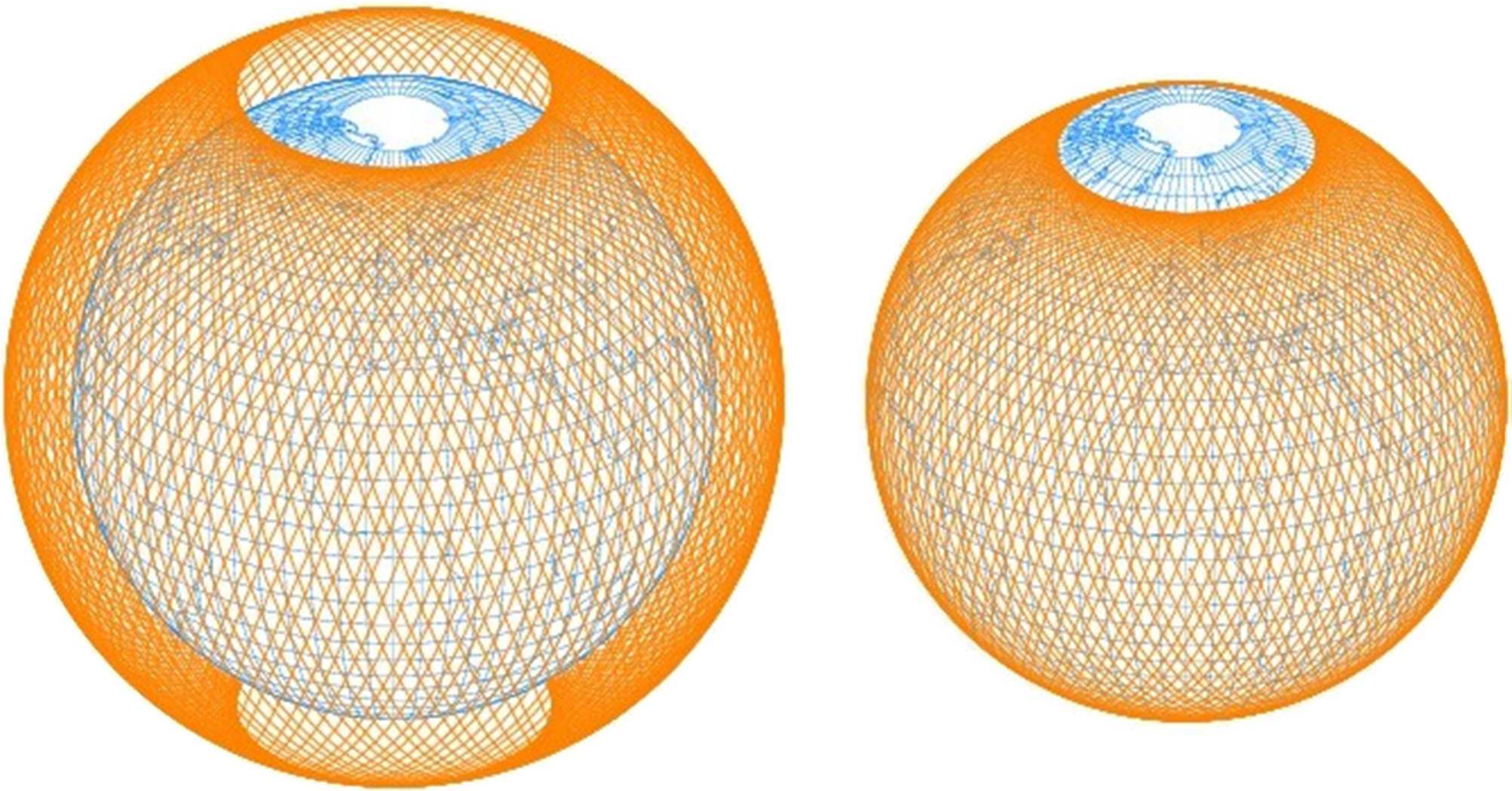
$$i = 98.7\ deg$$
$$i' = i + 4\ deg$$



GEO (lemniscate)

$$i' \approx 90\ deg + \frac{i}{2}$$

Orbit and track coverage



Jason: 1336 km, 66 deg. over 9.915 days

Orbit repeat cycle

Repetitive orbits wrt the Earth are obtained under the condition: $iT_{orb} = j\tau_{nod}$

where T_{orb} is the orbital period: $T_{orb} = \dfrac{2\pi}{\dot{M}+\dot{\omega}}$, τ_{nod} is the nodal period: $\tau_{nod} = \dfrac{2\pi}{\dot{\Omega}_E - \dot{\Omega}}$

i and *j* are integers which represent respectively the number of orbit revolutions and the number of Earth rotations.

Hence the repeat cycle condition is expressed by: $\boxed{\dfrac{i}{j} = \dfrac{\tau_{nod}}{T_{orb}} = \dfrac{\dot{M}+\dot{\omega}}{\dot{\Omega}_E - \dot{\Omega}}}$

Repeat period:

$$T_{rep} = j\tau_{nod} = iT_{orb}$$

$$= j\frac{2\pi}{\dot{\Omega}_E - \dot{\Omega}} = i\frac{2\pi}{\dot{M}+\dot{\omega}}$$

In case of **sunsynchronous** satellite:

$$\dot{\Omega} = \dot{\Omega}_S \quad \text{and} \quad \dot{\Omega}_E - \dot{\Omega}_S = \frac{2\pi}{86400}$$

$$\rightarrow \quad T_{rep} = j \quad (day)$$

	i orbit rev.	*j Earth rot.*	$T_{rep.}$ *(day)*
SEASAT	43	3	3
Jason	127	10	9.915
SPOT	369	26	26
ENVISAT	501	35	35

Orbit repeat cycle

Due to the Earth rotation, tracks precess westwards after each revolution by:

$$\boxed{\Delta\lambda = \left(\dot{\Omega}_E - \dot{\Omega}\right) T_{orb} = \frac{j}{i} 2\pi} \qquad \left(T_{orb} = \frac{2\pi}{\dot{M} + \dot{\omega}} \right)$$

A full rotation of the node in the Earth reference frame is accomplished after k revolutions when:

$$\left.\begin{array}{l} k\,\Delta\lambda \le 2\pi \\ (k+1)\Delta\lambda > 2\pi \end{array}\right\} \rightarrow \quad k \le \frac{2\pi}{\Delta\lambda} = \frac{i}{j} < k+1 \quad \Rightarrow \quad k = int\left(\frac{i}{j}\right) \quad , k\ integer$$

Then the comb of the next k ascending nodes is shifted:

after one day $\quad \Delta_1 = (k+1)\Delta\lambda - 2\pi = \left(\left(int\left(\frac{i}{j}\right)+1\right)\frac{j}{i} - 1\right)2\pi$

after 2 days $\quad \Delta_2 = (k_2+1)\Delta\lambda - 2(2\pi) = \left(\left(int\left(2\frac{i}{j}\right)+1\right)\frac{j}{i} - 2\right)2\pi$

after 3 days $\quad \Delta_3 = (k_3+1)\Delta\lambda - 3(2\pi) = \left(\left(int\left(3\frac{i}{j}\right)+1\right)\frac{j}{i} - 3\right)2\pi$

and so on…

Orbit repeat cycle

Inter-track distance

after n^+ days: $$R\Delta_{n^+} = \left((k_n+1)\Delta\lambda - n(2\pi)\right)R = \left(\left(\mathrm{int}\left(n\frac{i}{j}\right)+1\right)\frac{j}{i} - n\right)2\pi R$$

westwards (+)

or after n^- days: $$R\Delta_{n^-} = \left(k_n\,\Delta\lambda - n(2\pi)\right)R = \left(\mathrm{int}\left(n\frac{i}{j}\right)\frac{j}{i} - n\right)2\pi R$$ *(R: Earth radius)*

eastwards (-)

Closest tracks are when: $$\boxed{R\Delta_{n^\pm} = \left(n\,\mathrm{int}\left(n\frac{i}{j}\right)\frac{j}{i} - n\right)2\pi R}$$

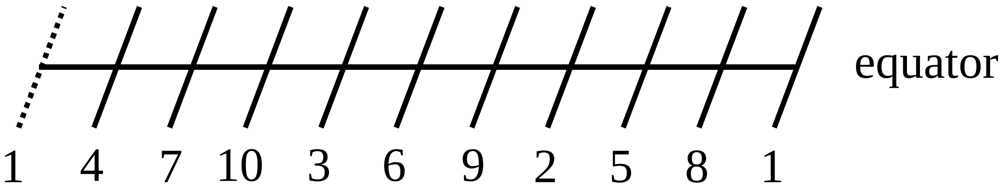
(nearest integer)

	i orbit rev.	*j Earth rot.*	*i/j*	*$R\Delta\lambda_{rev.}$ (km)*	*$R\Delta_{1\text{-}day}$ (km)*	*$n^\pm$ / $\Delta_{n\pm}$=min*	*$R\Delta_{min}$ (km)*
SEASAT	43	3	14.33	2793	-931	-1/2	931
Jason	127	10	12.7	3152	945	-3/7	315
SPOT	369	26	14.19	2820	-542	-21/5	108
ENVISAT	501	35	14.31	2796	-879	-16/19	80

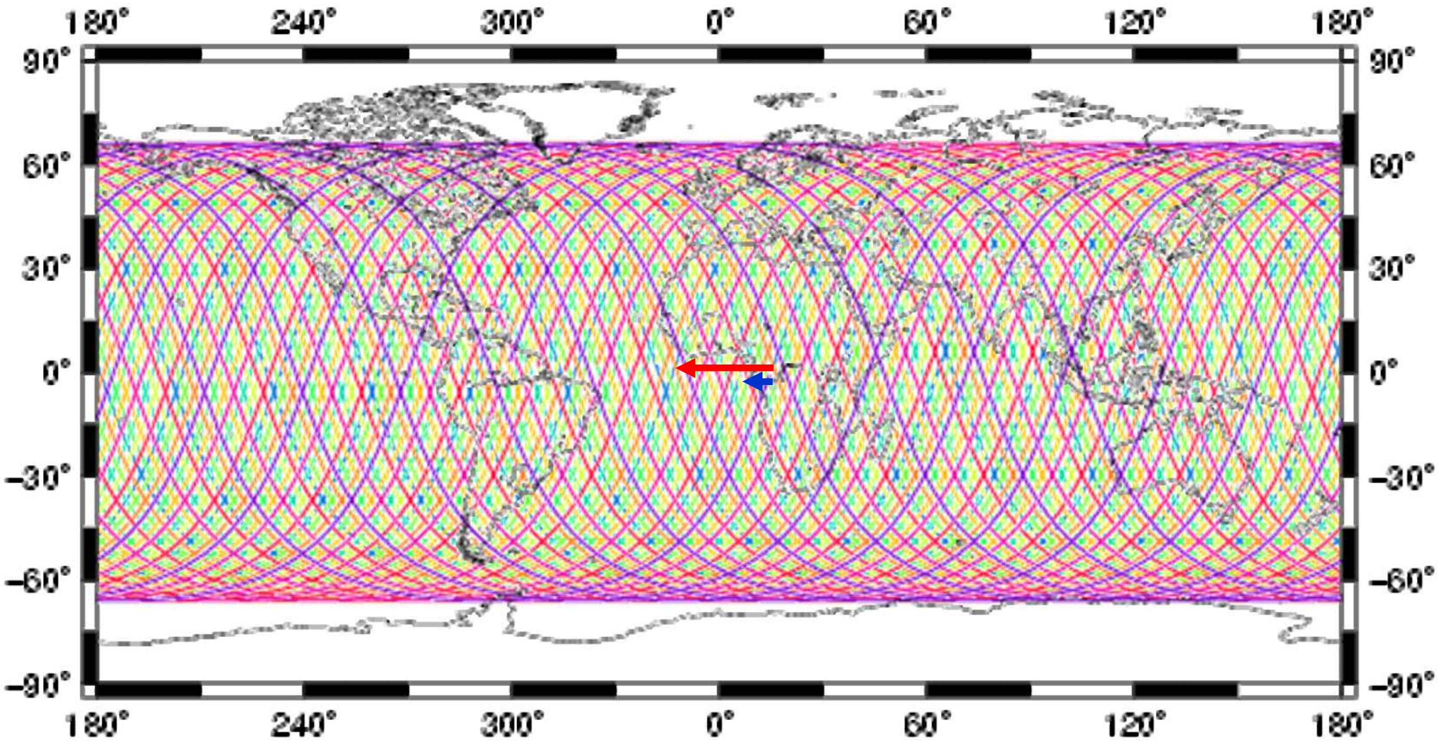
Jason sub-cycles

Shift after one rev.:
$R\Delta\lambda = 3150$ *km*

Shift after one day:
$R\Delta_1 = 945$ *km*



Shift after 3 days:
$R\Delta_{-3} = 315$ *km*



Sub-cycle

Tracks precessing regularly westwards describe an homogeneous grid over the full repeat cycle. However sub-cycles can appear, showing an homogeneous pattern as well, but with broader track distribution.

The number of these sub-cycles depends on the shift of the ascending node in the Earth reference frame. It can be inferred considering the following triplet (k, Δn_{rev}, j) such as :

$$\frac{i}{j} = \frac{\tau_{nod}}{T_{orb}} = \frac{\dot{M} + \dot{\omega}}{\dot{\Omega}_E - \dot{\Omega}} = k + \frac{\Delta n_{rev}}{j} \quad (rev./\,day) \quad with \quad k = n\,int\left(\frac{i}{j}\right)$$

k: nearest integer ratio between the number of revolutions and the number of Earth rotations

Jason : $\frac{i}{j} = \frac{127}{10} = 13 - \frac{3}{10}$; $R\Delta_{-3} = 315\,km$; 3-day sub-cycle ($R\Delta_1 = 945$ km)

SPOT : $\frac{i}{j} = \frac{369}{26} = 14 + \frac{5}{26}$; $R\Delta_5 = 108\,km$; 5-day sub-cycle ($R\Delta_1 = 542$ km)

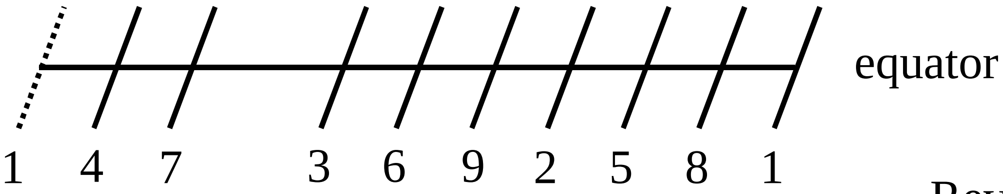
ENVISAT : $\frac{i}{j} = \frac{501}{35} = 14 + \frac{11}{35}$; $R\Delta_3 = 160\,km$; 3-day sub-cycle ($R\Delta_1 = 879$ km)

n* 11/35 ≈ 1

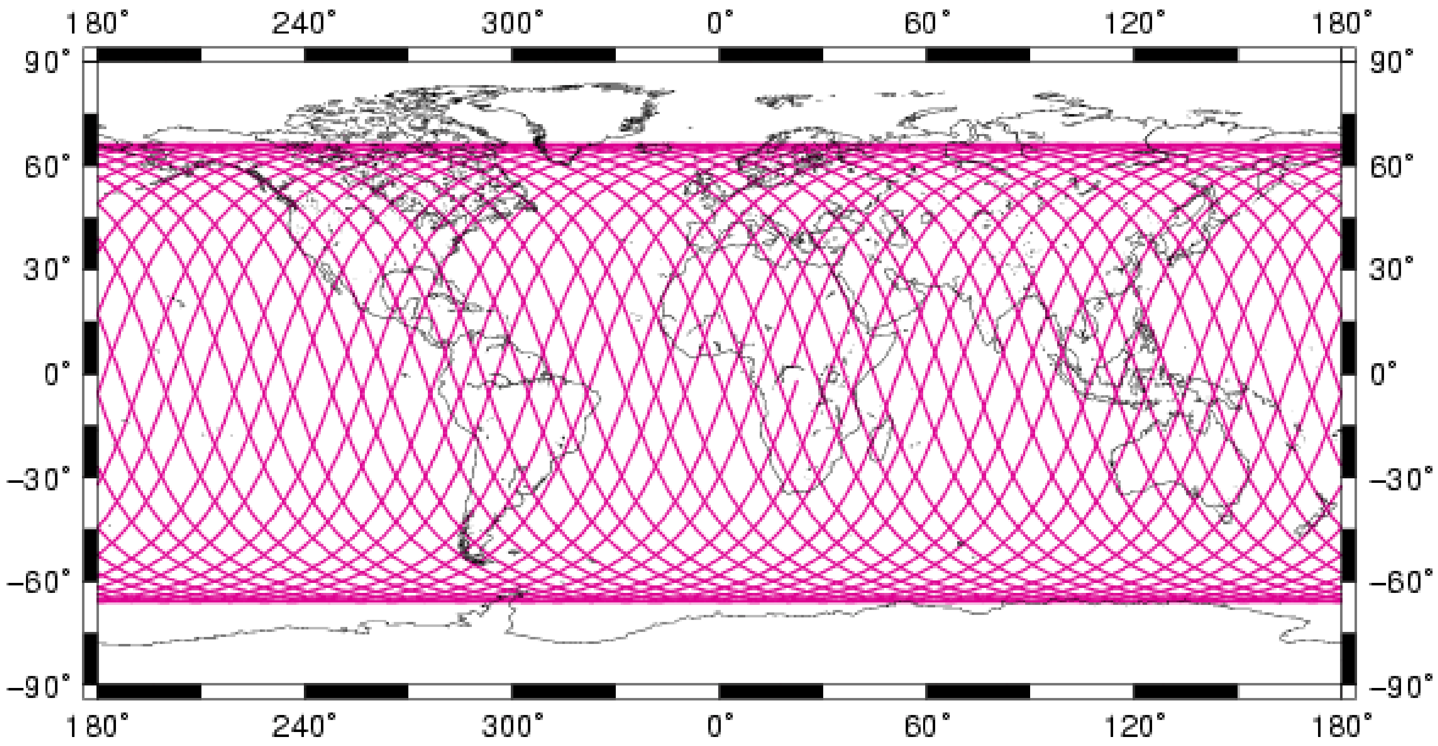
Jason sub-cycles

Shift after one rev.:
Δλ=3150 km

Inter-track distance
at equator: 945 km



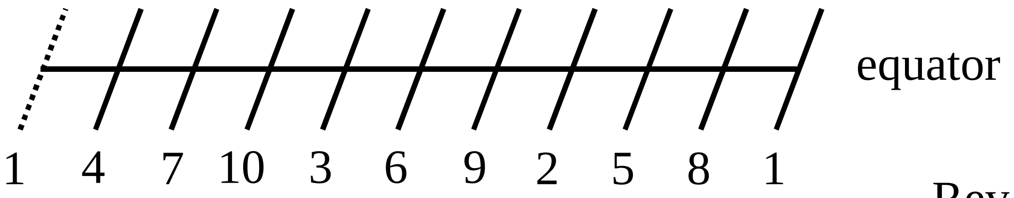
Rev. 77 - 114



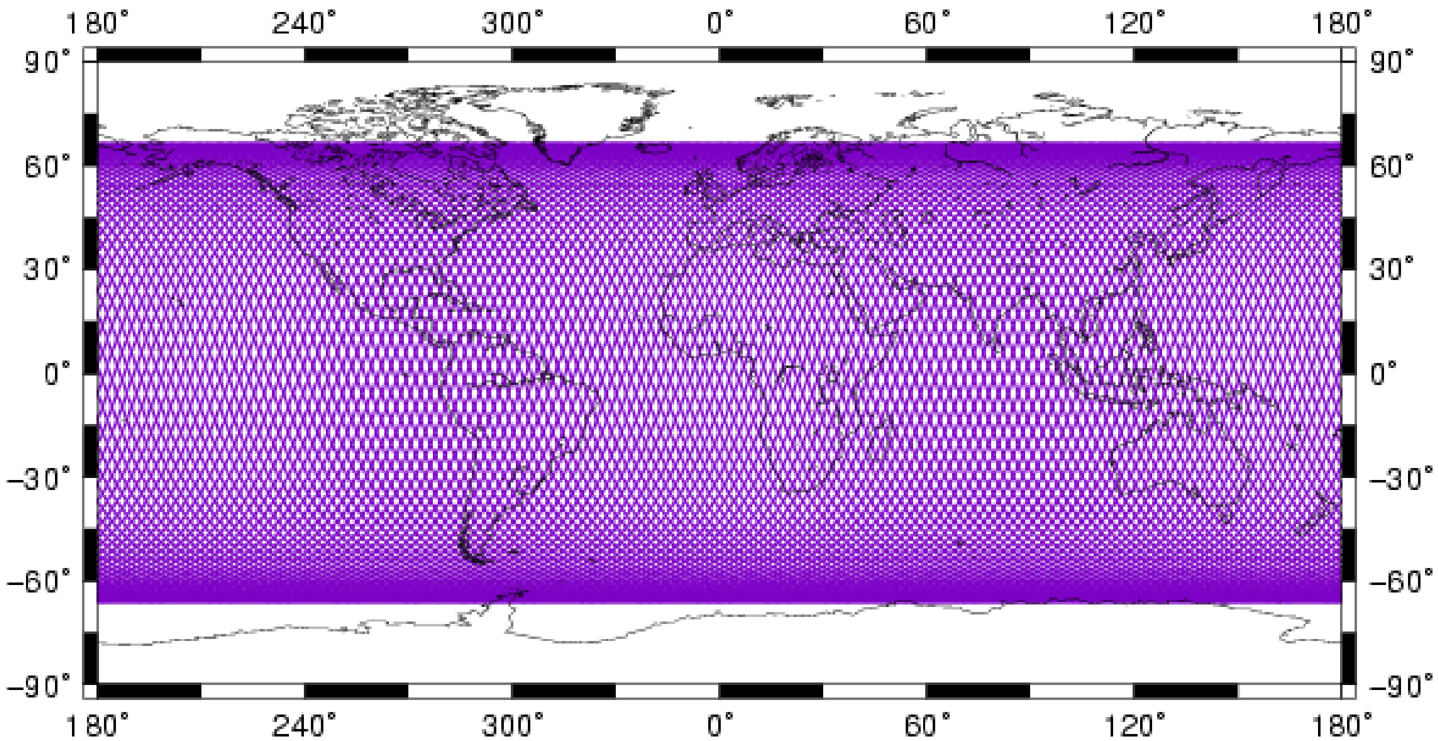
Jason sub-cycles

Shift after one rev.:
Δλ=3150 km

Inter-track distance
at equator: 315 km

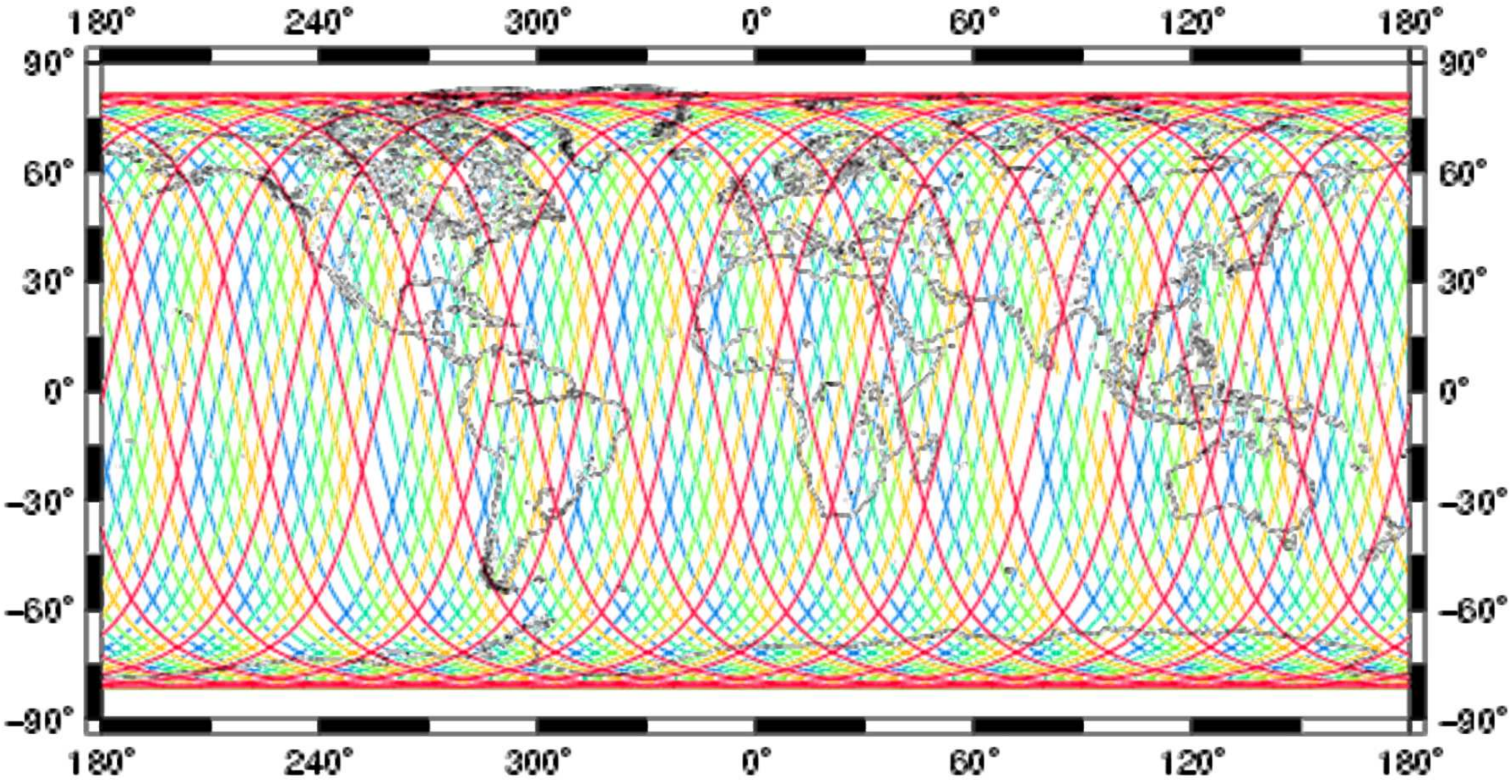


Rev. 1 - 127



SPOT sub-cycles

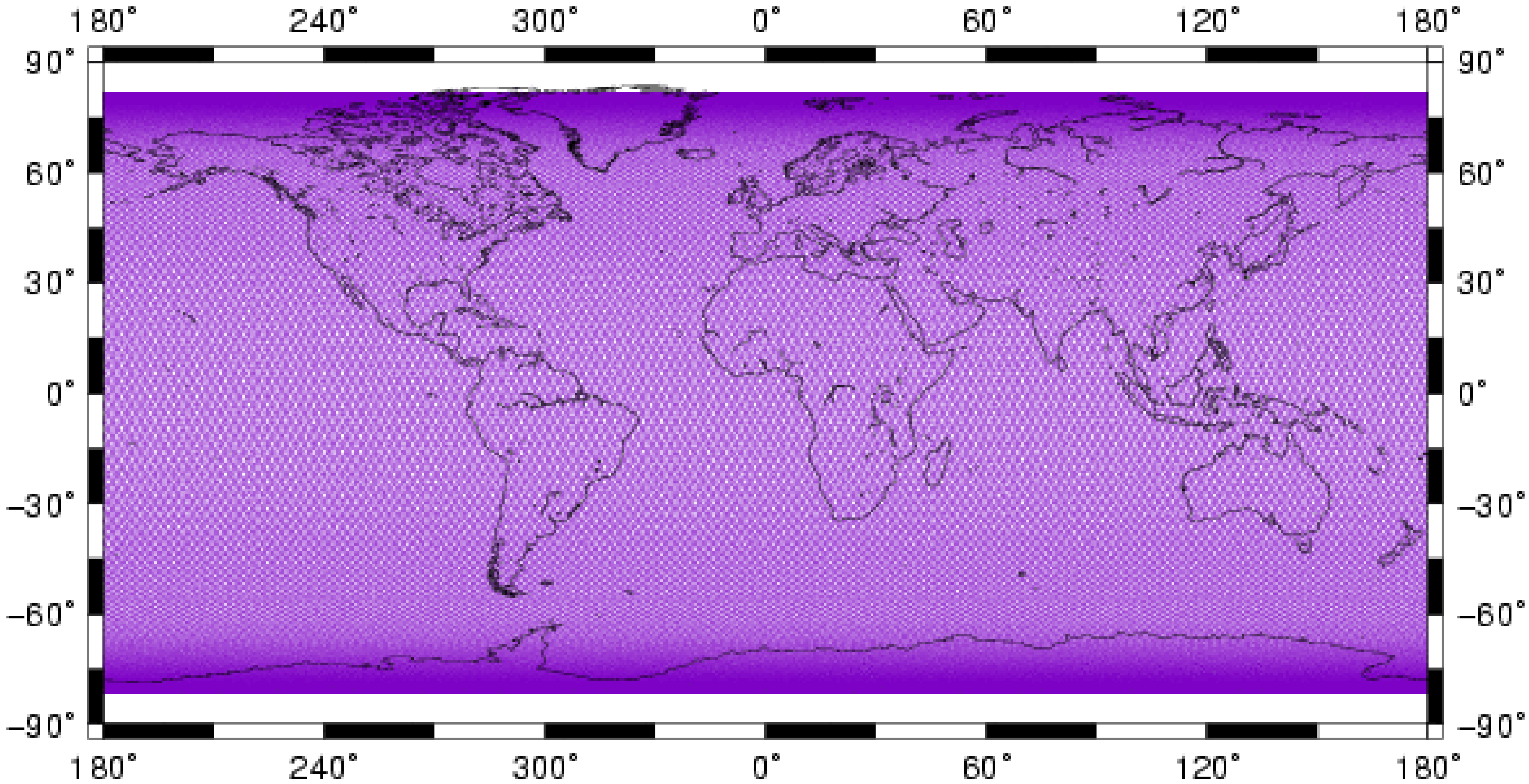
Revolutions 1 - 14 - 28 - 42 - 56 - 70



Shift after one rev. at the equator: 2820 km westwards, shift after one day: 542 km eastwards

SPOT cycle

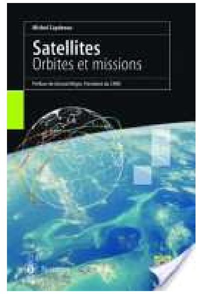
Revolutions 1 - 71 - 142 - 213 - 284 - 355 - 369

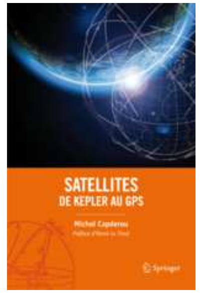


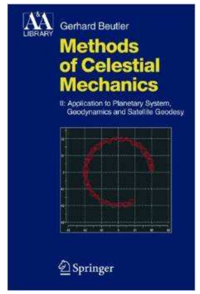
Inter-track distance at the equator after 5 days: 542 km, after 26 days: 108 km

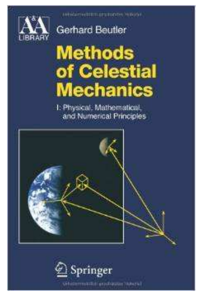
References

- Algorithmic Documentation of the GINS Software, Rapport GRGS, 2013, http://grgs.obs-mip.fr/recherche/logiciels
- Satellites: orbites et missions, M. Capderou, Springer, 2002
- Satellites: de Kepler au GPS, Springer, 2012
- Methods of Celestial Mechanics: Volume I/II: Physical, Mathematical, and Numerical Principles , G. Beutler, 2010
- Pseudo-Stochastic Orbit Modeling of Low Earth Satellites Using the Global Positioning System, A. Jäggi, *Geodätisch-geophysikalische Arbeiten in der Schweiz*, 2007, http://www.sgc.ethz.ch/sgc-volumes/sgk-73.pdf









Geographically correlated error

Radial orbit errors from gravity field covariance model display geographical patterns when expressed in geographic coordinates (φ, λ).
According to the Kaula's expansion, radial orbit errors are expressed in function of keplerian elements such as:

$$\Delta r = \sum_{l=2}^{L} \sum_{m=0}^{l} \sum_{p=0}^{l} C^{r}_{lmp0} S_{lmp0} \qquad with \qquad C^{r}_{lmp0} = na\left(\frac{a_e}{a}\right)^{l} F_{lmp}(i)\left[\frac{2(l-2p)}{\dot{\psi}_{lmp0}} + \frac{4p-3l-1}{2\dot{\psi}_{lmp1}} + \frac{4p-l+1}{2\dot{\psi}_{lmp-1}}\right]$$

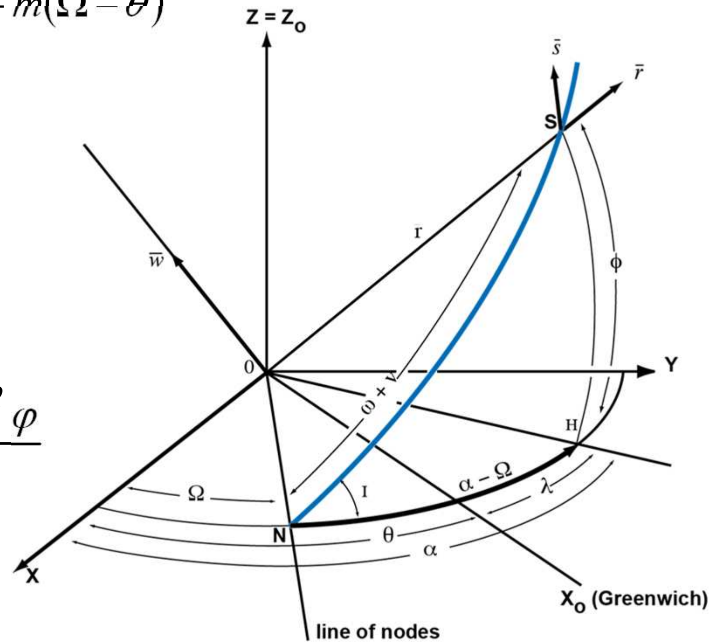
$$S_{lmp0} = \begin{Bmatrix} C_{lm} \\ -S_{lm} \end{Bmatrix}_{l-m\ impair}^{l-m\ pair} \cos\psi_{lmp0} + \begin{Bmatrix} S_{lm} \\ C_{lm} \end{Bmatrix}_{l-m\ impair}^{l-m\ pair} \sin\psi_{lmp0}$$

$$\psi_{lmp0} = (l-2p)(\omega + M) + m(\Omega - \theta)$$

Keplerian elements can then be converted into geographic coordinates using the following transformation formulas of spherical trigonometry:

$$\begin{cases} \sin\varphi = \sin i \ \sin(\omega + M) \\ \cos\varphi \cos(\alpha - \Omega) = \cos(\omega + M) \\ \cos\varphi \sin(\alpha - \Omega) = \cos i \ \sin(\omega + M) \end{cases} \Rightarrow \begin{cases} \sin(\omega + M) = \dfrac{\sin\varphi}{\sin i} \\ \cos^2(\omega + M) = \dfrac{\sin^2 i - \sin^2\varphi}{\sin^2 i} \end{cases}$$

$$\Omega - \theta = \lambda - (\alpha - \Omega)$$



Radial perturbations are then expressed in geographical coordinates such as :

$$\Delta r = \sum_{l=2}^{\infty}\sum_{m=0}^{l} M_{lm}\left(C_{lm}\cos m\lambda + S_{lm}\sin m\lambda\right) \qquad mean\ part$$

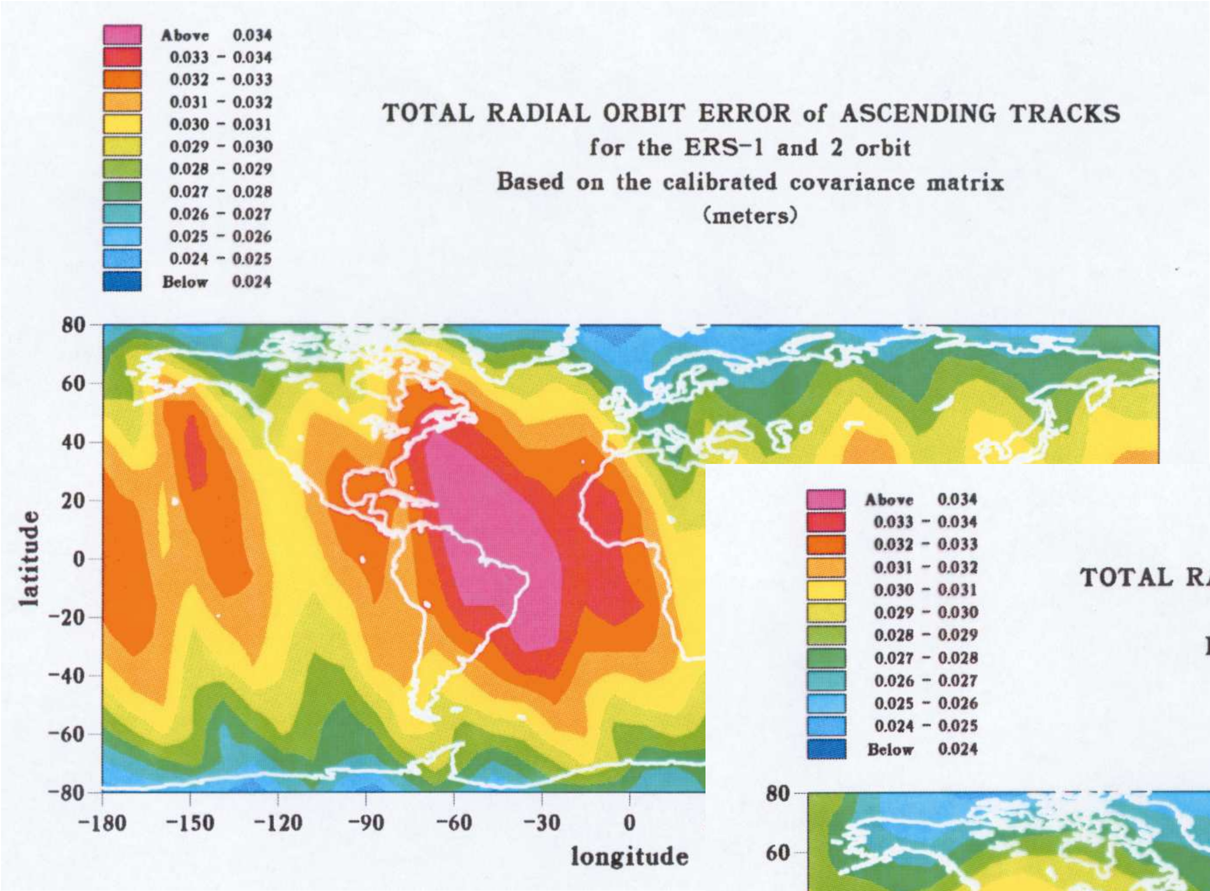
$$+(-1)^{\sigma}\sum_{l=2}^{\infty}\sum_{m=0}^{l} V_{lm}\left(C_{lm}\sin m\lambda - S_{lm}\cos m\lambda\right) \qquad var\,iable\ part$$

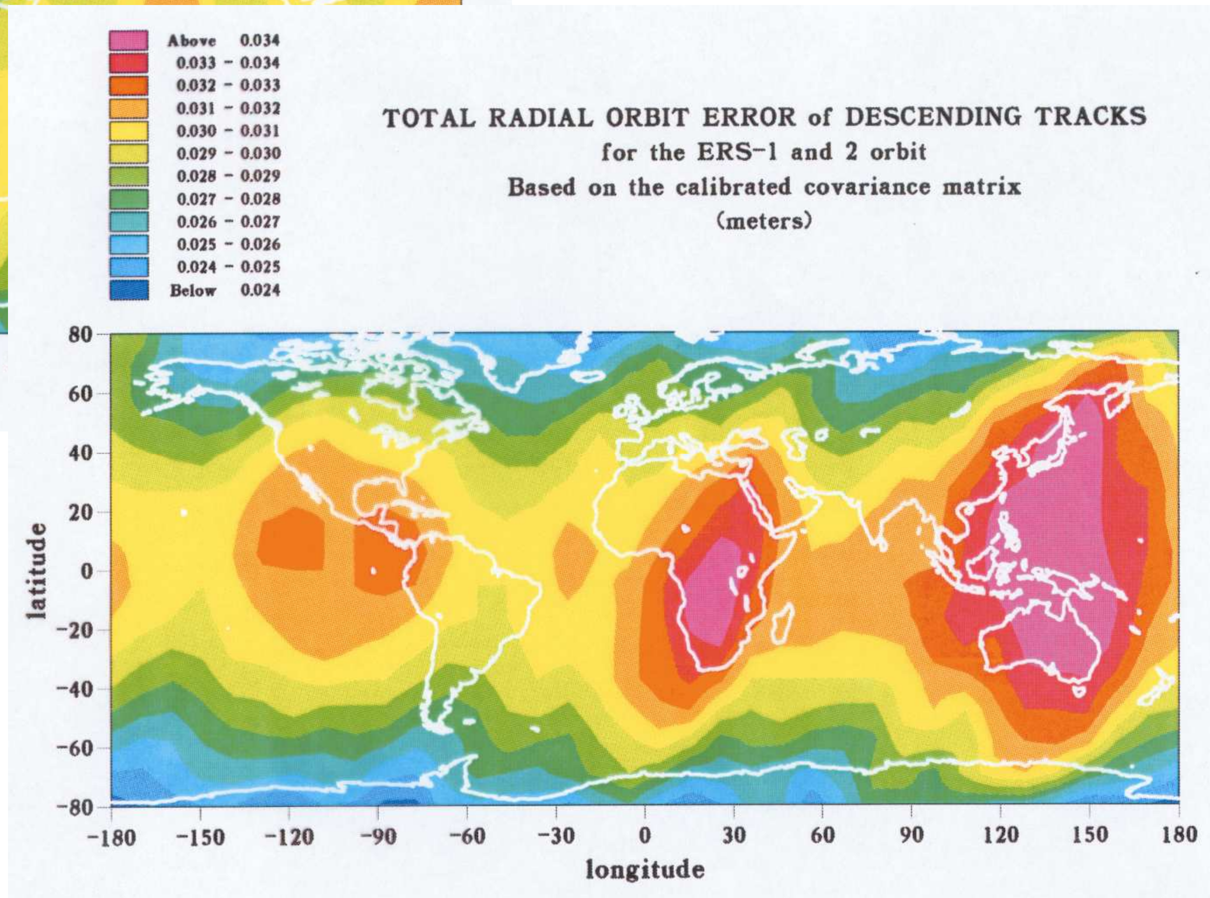
$$with: \quad M_{lm} = \sum_{p=0}^{l} C^{r}_{lmp0}\begin{pmatrix}\Phi^{c}_{lmp}\\ -\Phi^{c}_{lmp}\end{pmatrix}^{l-m\ pair}_{l-m\ impair} \qquad et \quad V_{lm} = \sum_{p=0}^{l} C^{r}_{lmp0}\begin{pmatrix}\Phi^{s}_{lmp}\\ \Phi^{c}_{lmp}\end{pmatrix}^{l-m\ pair}_{l-m\ inpair}$$

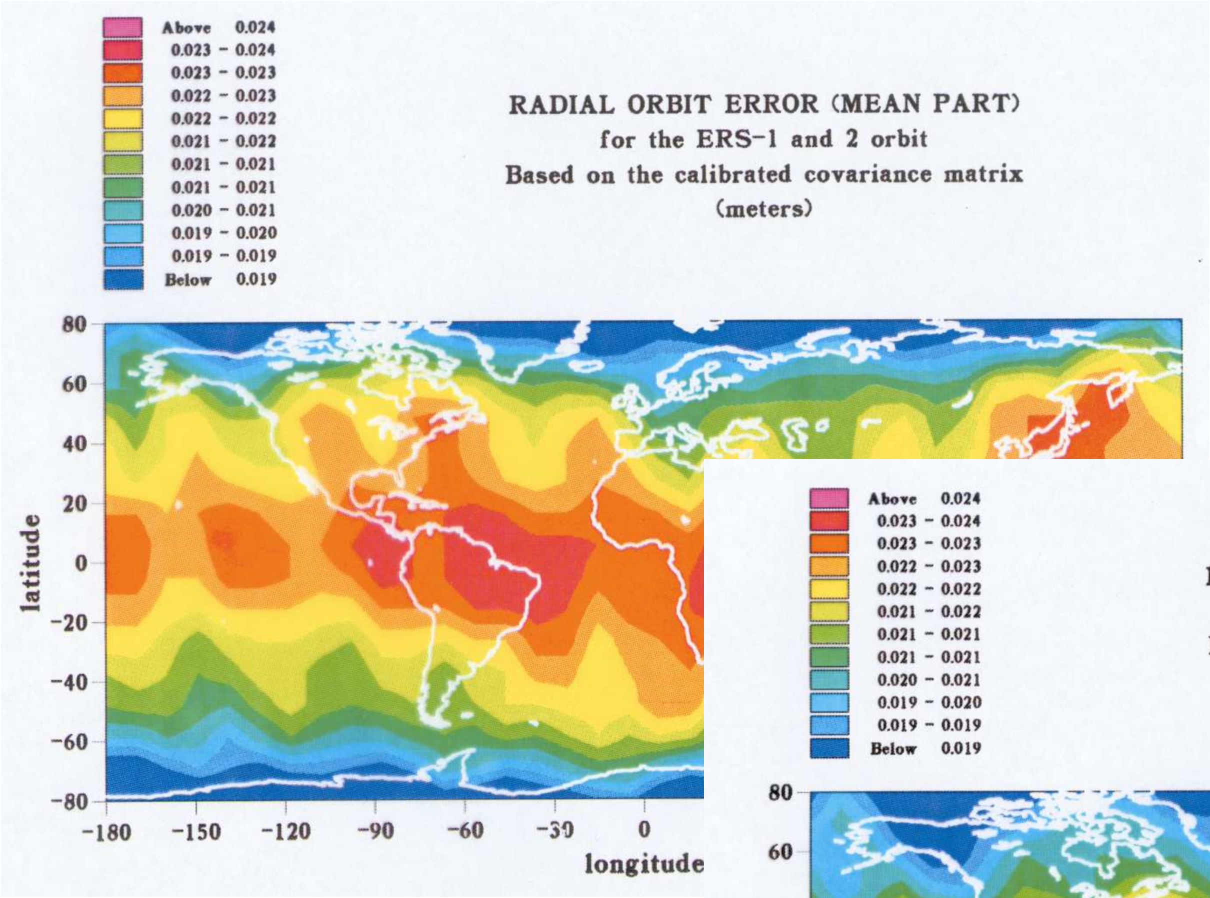
The sign indetermination on $\cos(\omega+M)$ lets distinguish between ascending passes ($\cos(\omega+M)>0$) and descending passes ($\cos(\omega+M)<0$) for which one defines respectively $\sigma=0$ and $\sigma=1$. Φ depends on latitude and inclination trigonometric functions.
Finally radial orbit errors give a synthetic quality information on a gravity field model from its covariance matrix Γ by forming the mean quadratic error :

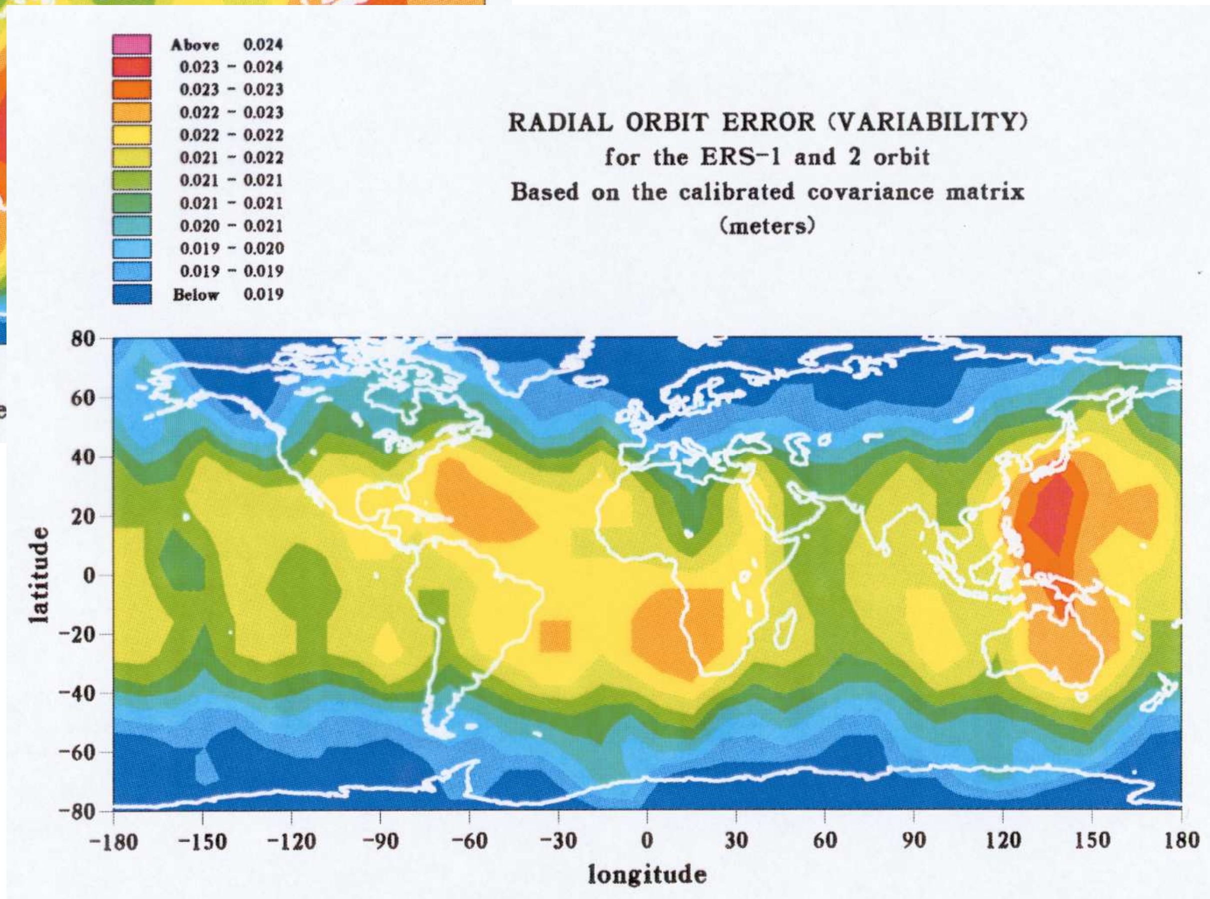
$$\begin{aligned}< E\left(\delta r^{2}_{lm}\right)> = \sigma^{2}(\delta r_{lm}) = M^{2}_{lm}&\left(\Gamma^{CC}_{lmlm}\cos^{2} m\lambda + \Gamma^{SS}_{lmlm}\sin^{2} m\lambda + 2\Gamma^{CS}_{lmlm}\cos m\lambda\sin m\lambda\right)\\ +V^{2}_{lm}&\left(\Gamma^{CC}_{lmlm}\sin^{2} m\lambda + \Gamma^{SS}_{lmlm}\cos^{2} m\lambda - 2\Gamma^{CS}_{lmlm}\cos m\lambda\sin m\lambda\right)\end{aligned}$$

$$\Gamma^{CC}_{lml'm'} = E\left[\delta C_{lm}\delta C_{l'm'}\right]; \quad \Gamma^{SS}_{lml'm'} = E\left[\delta S_{lm}\delta S_{l'm'}\right]; \quad \Gamma^{CS}_{lml'm'} = \Gamma^{SC}_{l'm'lm} = E\left[\delta C_{lm}\delta S_{l'm'}\right]$$

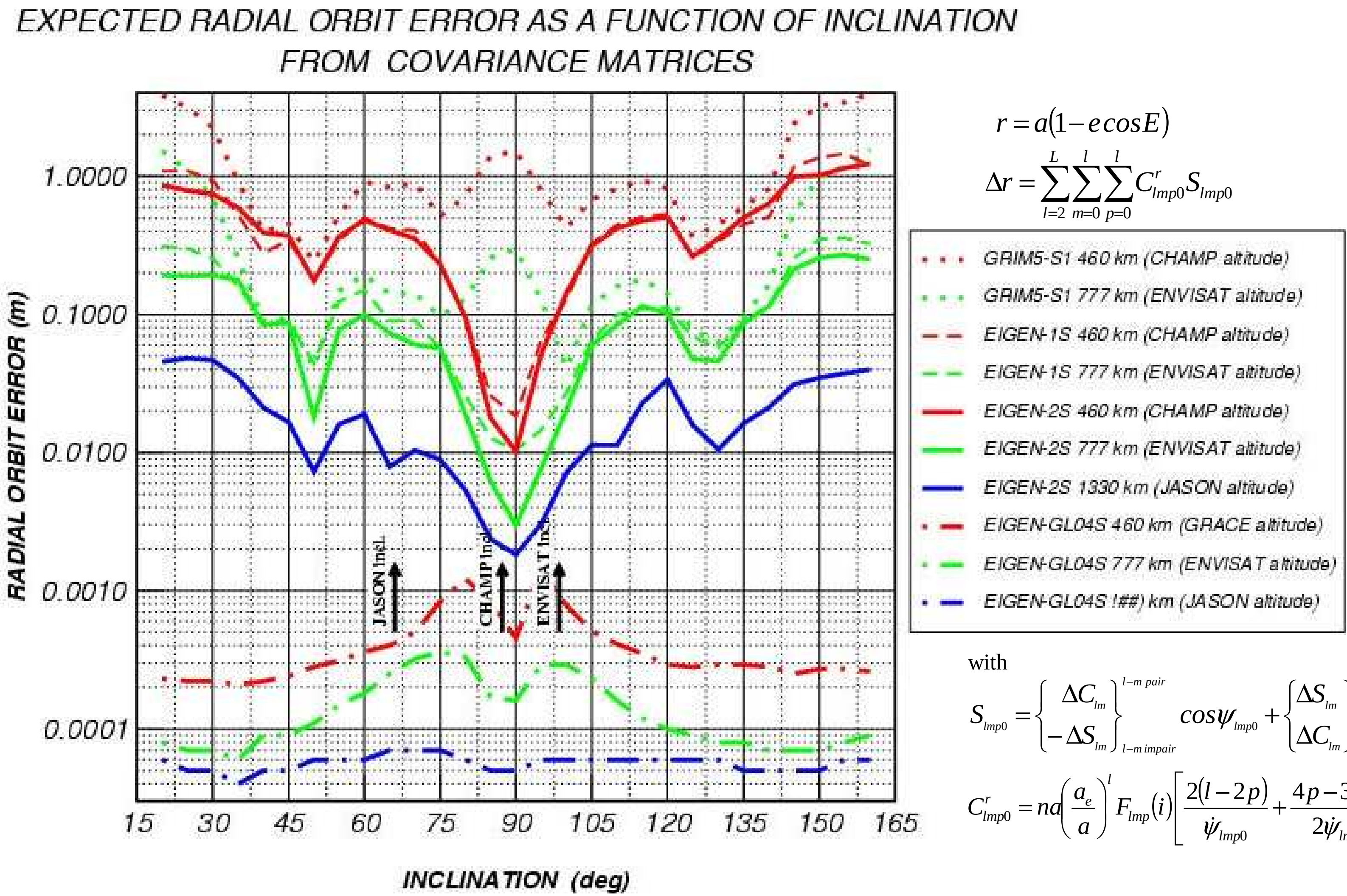

Above 0.034
0.033 - 0.034
0.032 - 0.033
0.031 - 0.032
0.030 - 0.031
0.029 - 0.030
0.028 - 0.029
0.027 - 0.028
0.026 - 0.027
0.025 - 0.026
0.024 - 0.025
Below 0.024
TOTAL RADIAL ORBIT ERROR of ASCENDING TRACKS
for the ERS-1 and 2 orbit
Based on the calibrated covariance matrix
(meters)
latitude
80
60
40
20
0
-20
-40
-60
-80
-180
-150
-120
-90
-60
-30
0
longitude


Above 0.034
0.033 - 0.034
0.032 - 0.033
0.031 - 0.032
0.030 - 0.031
0.029 - 0.030
0.028 - 0.029
0.027 - 0.028
0.026 - 0.027
0.025 - 0.026
0.024 - 0.025
Below 0.024
TOTAL RADIAL ORBIT ERROR of DESCENDING TRACKS
for the ERS-1 and 2 orbit
Based on the calibrated covariance matrix
(meters)
latitude
80
60
40
20
0
-20
-40
-60
-80
-180
-150
-120
-90
-60
-30
0
30
60
90
120
150
180
longitude


Above 0.024
0.023 - 0.024
0.023 - 0.023
0.022 - 0.023
0.022 - 0.022
0.021 - 0.022
0.021 - 0.021
0.021 - 0.021
0.020 - 0.021
0.019 - 0.020
0.019 - 0.019
Below 0.019
RADIAL ORBIT ERROR (MEAN PART)
for the ERS-1 and 2 orbit
Based on the calibrated covariance matrix
(meters)
latitude
longitude


Above 0.024
0.023 - 0.024
0.023 - 0.023
0.022 - 0.023
0.022 - 0.022
0.021 - 0.022
0.021 - 0.021
0.021 - 0.021
0.020 - 0.021
0.019 - 0.020
0.019 - 0.019
Below 0.019
RADIAL ORBIT ERROR (VARIABILITY)
for the ERS-1 and 2 orbit
Based on the calibrated covariance matrix
(meters)
latitude
longitude

Projected radial orbit errors from covariance matrix



$$r = a(1 - e\cos E)$$

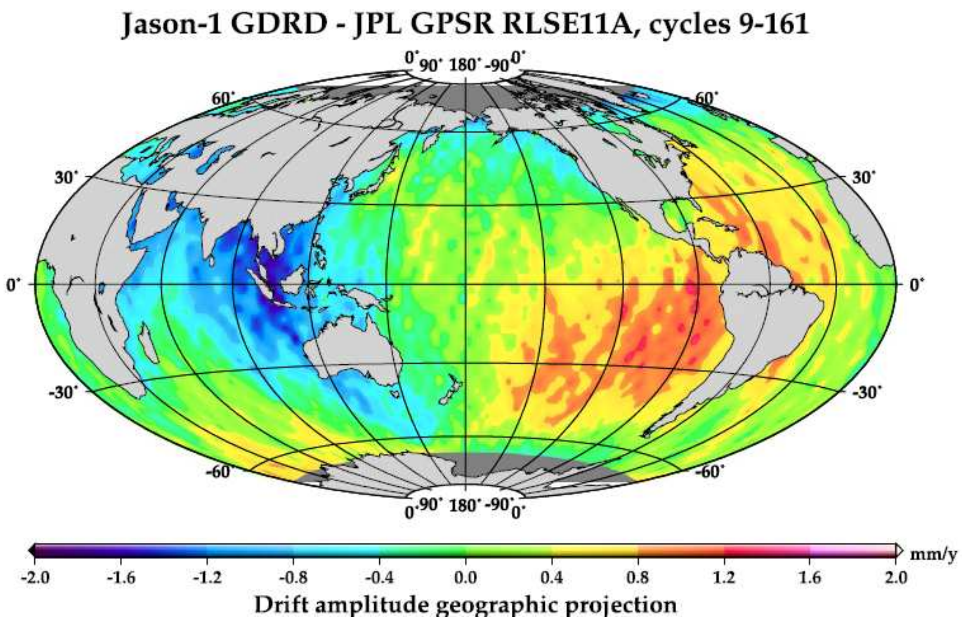
$$\Delta r = \sum_{l=2}^{L}\sum_{m=0}^{l}\sum_{p=0}^{l} C^r_{lmp0} S_{lmp0}$$

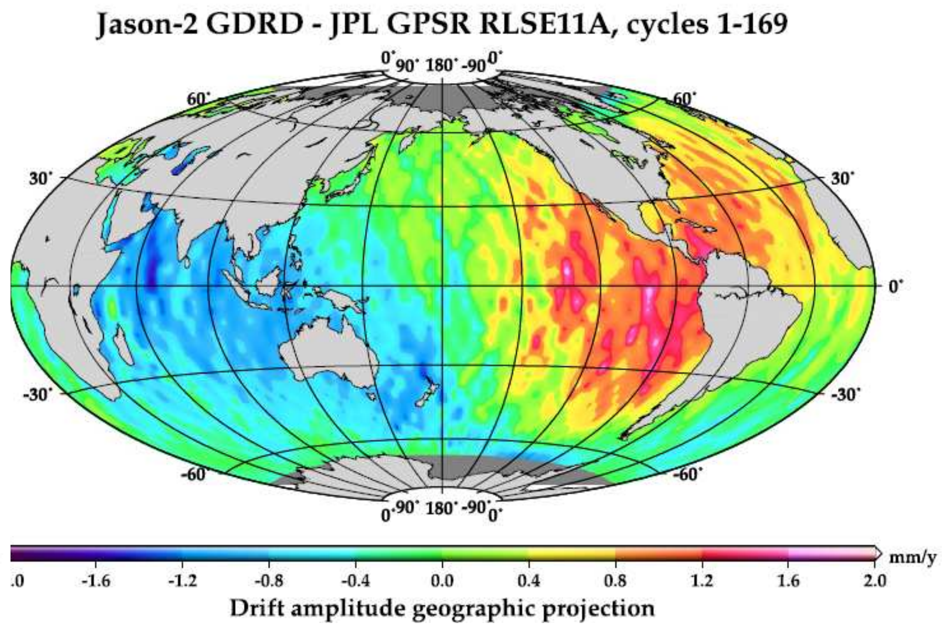
with

$$S_{lmp0} = \begin{Bmatrix} \Delta C_{lm} \\ -\Delta S_{lm} \end{Bmatrix}_{l-m\,impair}^{l-m\,pair} \cos\psi_{lmp0} + \begin{Bmatrix} \Delta S_{lm} \\ \Delta C_{lm} \end{Bmatrix}_{l-m\,impair}^{l-m\,pair} \sin\psi_{lmp0}$$

$$C^r_{lmp0} = na\left(\frac{a_e}{a}\right)^l F_{lmp}(i)\left[\frac{2(l-2p)}{\dot\psi_{lmp0}} + \frac{4p-3l-1}{2\dot\psi_{lmp1}} + \frac{4p-l+1}{2\dot\psi_{lmp-1}}\right]$$

Geographically correlated radial difference drifts
Comparison between dynamic and reduced dynamic orbits

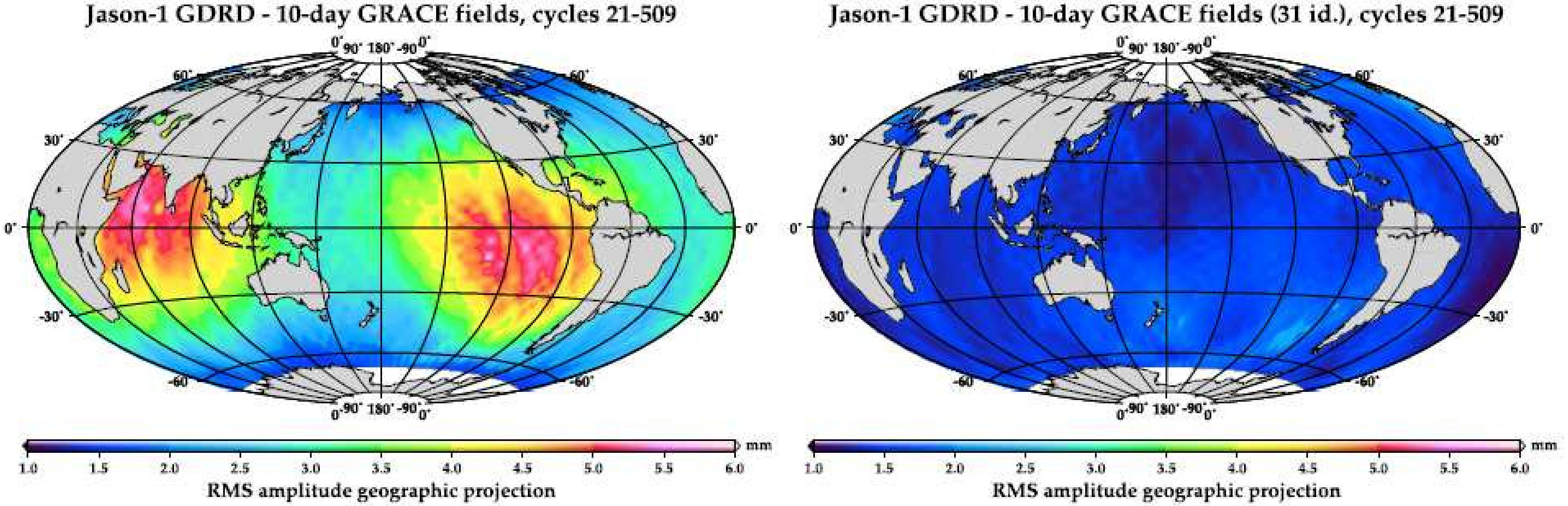




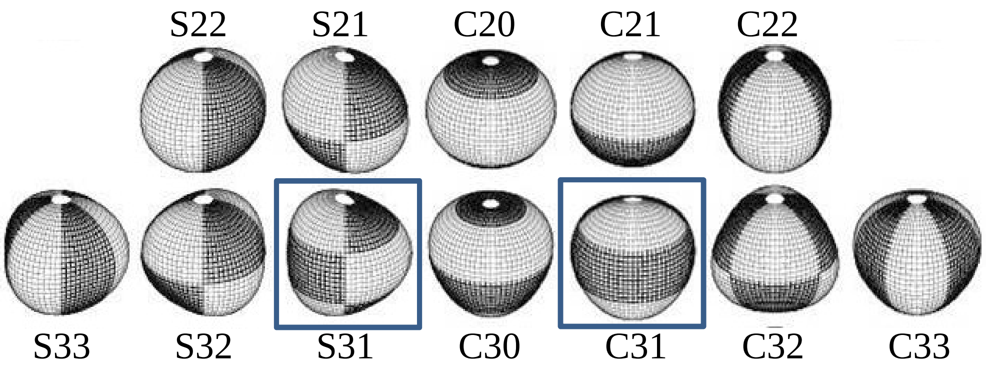
Jason-1 and Jason-2 geographically correlated radial difference drifts between GPS-Reduced-dynamic (right; cycles 9–161; April 2002–May 2006) solutions compared to the GDR-D orbit series, over 3.5 x3.5 bins.

A. Couhert et al. , Advances in Space Research (2014)

Impact of Jason radial orbit errors on the mean sea surface induced by C31/S31 gravity field coefficients

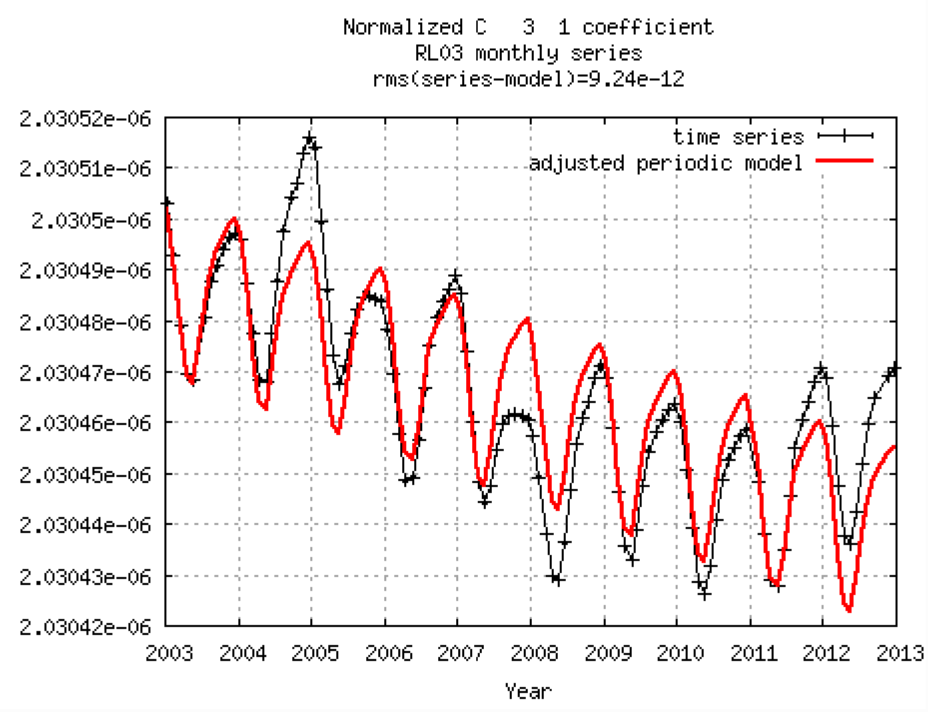


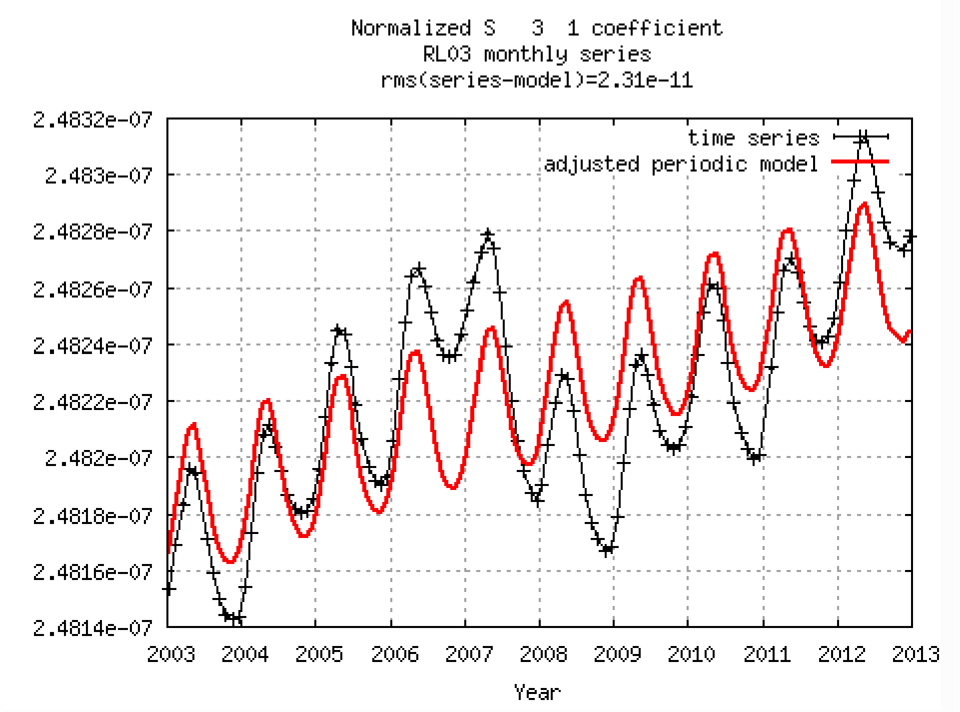
Jason-1 regional RMS radial differences (cycles 21–509; over 3:5x3:5 bins), between the DORIS-only GDR-D-like and 10-day fields orbit series (left) or 10-day fields orbit series making C31/S31 identical (right).



A. Couhert et al. , Advances in Space Research (2014)

C31/S31 time variable gravity field coefficients from GRACE/Lageos data





http://grgs.obs-mip.fr/grace/variable-models-grace-lageos/interactive-tools/Visualization-of-time-series-harmonic-coefficients

Transformation parameters from ITRF2008 to ITRF2005

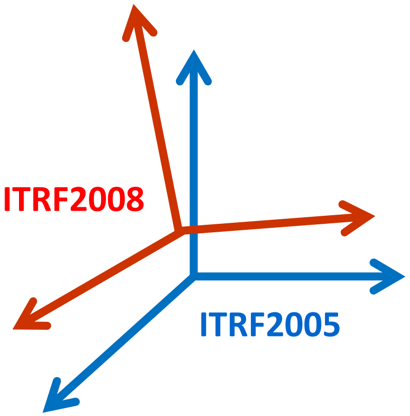
Table 6 Transformation Parameters at epoch 2005.0 and their rates from ITRF2008 to ITRF2005, to be used with Eq. 4

	Tx (mm) $\dot{T}x$ (mm/yr)	Ty (mm) $\dot{T}y$ (mm/yr)	Tz (mm) $\dot{T}z$ (mm/yr)	D (ppb) $\dot{D}$ (ppb/yr)	Rx (mas) $\dot{R}x$ (mas/yr)	Ry (mas) $\dot{R}y$ (mas/yr)	Rz (mas) $\dot{R}z$ (mas/yr)
	−0.5	−0.9	−4.7	0.94	0.00	0.00	0.00
±	0.2	0.2	0.2	0.03	0.08	0.08	0.08
	0.3	0.0	0.0	0.00	0.00	0.00	0.00
±	0.2	0.2	0.2	0.03	0.08	0.08	0.08

$$\begin{cases} \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{i05} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{i08} + T + D \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{i08} + R \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{i08} \\ \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}_{i05} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}_{i08} + \dot{T} + \dot{D} \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{i08} + \dot{R} \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{i08} \end{cases} \quad (4)$$

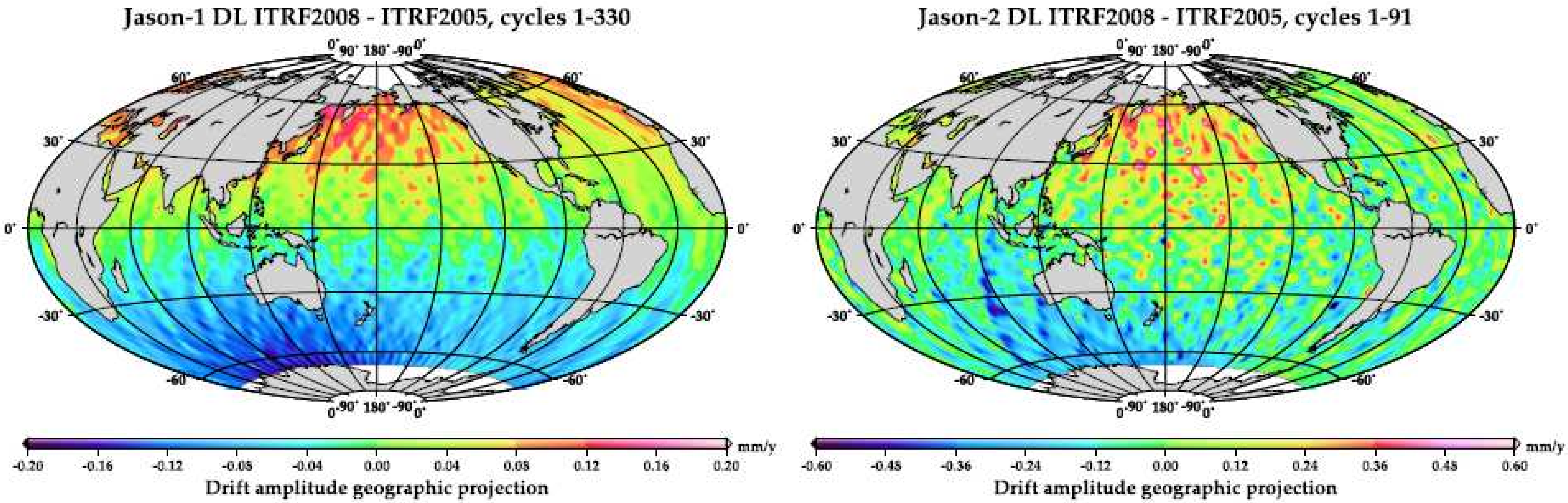
where $i05$ designates ITRF2005 and $i08$ ITRF2008, T is the translation vector, $T = (T_x, T_y, T_z)^T$, D is the scale factor and R is the matrix containing the rotation angles, given by

$$R = \begin{pmatrix} 0 & -R_z & R_y \\ R_z & 0 & -R_x \\ -R_y & R_x & 0 \end{pmatrix}$$



ITRF2008: an improved solution of the international terrestrial reference frame
Zuheir Altamimi, Xavier Collilieux, Laurent Métivier
Journal of Geodesy, 2011

Impact of Jason radial orbit errors on the mean sea surface induced by drift discrepancies in terrestrial reference frames



Jason-1 (left; cycles 1–330) and Jason-2 (right; cycles 1–91) DORIS + SLR ITRF2008 – ITRF2005 regional radial orbit trend differences over 3.5x3.5 bins.

Discrepancies between both networks result in a drift of **0.15±0.02 mm/yr and 0.26 ±0.11 mm/yr in Z for the Jason-1 and Jason-2 orbits**. Such an **orbit Z-shift** is mirrored in the Jason-1 and Jason-2 global MSL observations (because of the asymmetric distribution of the oceans between the northern and southern hemispheres), consistently with the transfer function : **Global MSL error = 0.16 ΔZ**

A. Couhert et al. , Advances in Space Research (2014)


NASA's Lunar Reconnaissance Orbiter (LRO), 2009-2014
the end