CATREF Software
Combination and Analysis of Terrestrial Reference Frames

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Contents

Introduction: History

- **First ITRF (BTS) combination Software was designed in the 80's**
  - **Mean station positions + EOP (diagonal terms)**
  - **No velocity estimates**
  - **AM02 plate motion model**

- **2nd ITRF Combination Software built end 80's using two steps:**
  - **Combination of station positions with full variances**
  - **Combination of station velocities with full variances**
  - **EOP's were not included**

- **CATREF project started in Nov. 1995 with the advent of SINEX format:**
  - **Reads and writes SINEX with full variances**
  - **Station positions & velocities**
  - **EOP added in 2001**
  - **Well adapted for time series combination (rigorously stacking)**
  - **Handles constraints before combination**

General Transformation between two TRS (1/2)

7-parameter similarity :

$$\boxed{X_2 = T + \lambda.\mathcal{R}.X_1}$$

Translation Vector $T = (T_x, T_y, T_z)^T$

Scale Factor λ

Rotation Matrix $\mathcal{R} = R_x.R_y.R_z$

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos R1 & \sin R1 \\ 0 & -\sin R1 & \cos R1 \end{pmatrix}$$

$$R_y = \begin{pmatrix} \cos R2 & 0 & -\sin R2 \\ 0 & 1 & 0 \\ \sin R2 & 0 & \cos R2 \end{pmatrix}$$

$$R_z = \begin{pmatrix} \cos R3 & \sin R3 & 0 \\ -\sin R3 & \cos R3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

General Transformation between two TRS (2/2)

In space geodesy we use the linearized formula:

$$X_2 = X_1 + T + DX_1 + R.X_1 \tag{1}$$

with: $T = (T_x, T_y, T_z)^T$**,** $\lambda = (1 + D)$**, and** $\mathcal{R} = (I + R)$

where $R = \begin{pmatrix} 0 & -R3 & R2 \\ R3 & 0 & -R1 \\ -R2 & R1 & 0 \end{pmatrix}$

since T **is less than 100 meters,** D **&** R **less than** 10^{-5}

The terms of 2nd ordre are neglected: less than $10^{-10} \approx 0.6$ **mm.**

Differentiating equation 1 with respect to time, we have:

$$\dot{X}_2 = \dot{X}_1 + \dot{T} + \overbrace{D\dot{X}_1}^{\approx 0} + \dot{D}X_1 + \overbrace{R\dot{X}_1}^{\approx 0} + \dot{R}X_1 \tag{2}$$

Combination Model: Inputs/Outputs

We assume given (Inputs):

- **a network of N points:** $x_0^i, y_0^i, z_0^i, \dot{x}_0^i, \dot{y}_0^i, \dot{z}_0^i$, $i = 1, ..., N$
- K **TRF**
- **a set of S solutions with :**
  - **station positions (& velocities)** x_s^i, y_s^i, z_s^i, $\dot{x}_s^i$, $\dot{y}_s^i$, $\dot{z}_s^i$, t_s^i, $s = 1, ..., S$
  - **polar motion:** x_s^p, y_s^p **and** UT_s **and their daily rates:** $\dot{x}_s^p$, $\dot{y}_s^p$, LOD_s
  - **with full variance covariance matrix:** $\sum_s$

The unknowns (Outputs) are:

- **Positions at t_0 and velocities of the N points:** $x^i, y^i, z^i, \dot{x}^i, \dot{y}^i, \dot{z}^i$ **expressed in the combined frame,**
- **The 7 transformation parameters at t_k and their time derivatives between each TRF k and the combined TRF**
- L **pole parameters:** x^p, y^p, UT, $\dot{x}^p$, $\dot{y}^p$, LOD

Combination Model: basic equations

$$\left\{\begin{aligned} \begin{pmatrix} x_s^i \\ y_s^i \\ z_s^i \end{pmatrix} &= \begin{pmatrix} x^i \\ y^i \\ z^i \end{pmatrix} + (t_s^i - t_0)\begin{pmatrix} \dot{x}^i \\ \dot{y}^i \\ \dot{z}^i \end{pmatrix} + T_k + D_k \begin{pmatrix} x^i \\ y^i \\ z^i \end{pmatrix} + R_k \begin{pmatrix} x^i \\ y^i \\ z^i \end{pmatrix} \\ &+ (t_s^i - t_k)\left[\dot{T}_k + \dot{D}_k \begin{pmatrix} x^i \\ y^i \\ z^i \end{pmatrix} + \dot{R}_k \begin{pmatrix} x^i \\ y^i \\ z^i \end{pmatrix}\right] \\ \\ \begin{pmatrix} \dot{x}_s^i \\ \dot{y}_s^i \\ \dot{z}_s^i \end{pmatrix} &= \begin{pmatrix} \dot{x}^i \\ \dot{y}^i \\ \dot{z}^i \end{pmatrix} + \dot{T}_k + \dot{D}_k \begin{pmatrix} x^i \\ y^i \\ z^i \end{pmatrix} + \dot{R}_k \begin{pmatrix} x^i \\ y^i \\ z^i \end{pmatrix} \end{aligned}\right. \qquad (3)$$

$$\left\{\begin{aligned} x_s^p &= x^p + R2_k \\ y_s^p &= y^p + R1_k \\ UT_s &= UT - \tfrac{1}{f} R3_k \\ \dot{x}_s^p &= \dot{x}^p + \dot{R}2_k \\ \dot{y}_s^p &= \dot{y}^p + \dot{R}1_k \\ LOD_s &= LOD + \tfrac{\Lambda_0}{f} \dot{R}3_k \end{aligned}\right. \qquad (4)$$

CATREF Units

- **meter for positions and meter/year for velocities**
- **meter for translations and meter/year for their rates**
- 10^{-6} **for the scale and** 10^{-6} **/year for scale rate**
- **arc-seconds for rotations and arc-seconds/year for their rates**
- mas **for polar motion and** mas**/day for their rates**
- ms **for** UT **and** ms**/day for** LOD**.**

X & $\dot{X}$ observation equations

$$\left\{\begin{array}{lcl}
x_s^i & = & x^i + dt_s^i \dot{x}^i + T1_k + x^i D_k 10^{-6} - cR3_k y^i + cR2_k z^i \\
& + & dt_k^i [\dot{T}1_k + x^i \dot{D}_k 10^{-6} - c\dot{R}3_k y^i + c\dot{R}2_k z^i] \\
\\
y_s^i & = & y^i + dt_s^i \dot{y}^i + T2_k + y^i D_k 10^{-6} + cR3_k x^i - cR1_k z^i \\
& + & dt_k^i [\dot{T}2_k + y^i \dot{D}_k 10^{-6} + c\dot{R}3_k x^i - c\dot{R}1_k z^i] \\
\\
z_s^i & = & z^i + dt_s^i \dot{z}^i + T3_k + z^i D_k 10^{-6} + cR1_k y^i - cR2_k x^i \\
& + & dt_k^i [\dot{T}3_k + z^i \dot{D}_k 10^{-6} + c\dot{R}1_k y^i - c\dot{R}2_k x^i] \\
\\
\dot{x}_s^i & = & \dot{x}^i + \dot{T}1_k + x^i \dot{D}_k 10^{-6} - c\dot{R}3_k y^i + c\dot{R}2_k z^i \\
\\
\dot{y}_s^i & = & \dot{y}^i + \dot{T}2_k + y^i \dot{D}_k 10^{-6} + c\dot{R}3_k x^i - c\dot{R}1_k z^i \\
\\
\dot{z}_s^i & = & \dot{z}^i + \dot{T}3_k + z^i \dot{D}_k 10^{-6} + c\dot{R}1_k y^i - c\dot{R}2_k x^i
\end{array}\right. \qquad (5)$$

with $dt_s^i = t_s^i - t_0$, $dt_k^i = t_s^i - t_k$ **and** $c = \frac{\pi}{180 \times 3600}$

EOP observation equations

$$\begin{cases} x_s^p(t_p) = x^p(t_p) + (R2_k + dt_k^p \frac{\dot{R}2_k}{y}) \times 1000 \\ y_s^p(t_p) = y^p(t_p) + (R1_k + dt_k^p \frac{\dot{R}1_k}{y}) \times 1000 \\ UT_s(t_p) = UT(t_p) - \frac{1}{f}(R3_k + dt_k^p \frac{\dot{R}3_k}{y}) \times \frac{1000}{15} \\ \dot{x}_s^p = \dot{x}^p + \frac{\dot{R}2_k}{y} \times 1000 \\ \dot{y}_s^p = \dot{y}^p + \frac{\dot{R}1_k}{y} \times 1000 \\ LOD_s = LOD + \frac{1}{f}\frac{\dot{R}3_k}{y} \times \frac{1000}{15} \end{cases} \tag{6}$$

with $dt_k^p = (t_p - t_k)$, $y = 365.25$ **and** $f = 1.002737909350795$

Linearized Unknowns

The unknown parameters are linearized around their approximate values: x_0^i**,** y_0^i**,** z_0^i**,** $\dot{x}_0^i$**,** $\dot{y}_0^i$**,** $\dot{z}_0^i$**,** x_0^p**,** y_0^p**,** $\dot{x}_0^p$**,** $\dot{y}_0^p$**,** UT_0 **and** LOD_0**, so that we have:**

$x^i = x_0^i + \delta x^i$ **(respectively** y^i**,** z^i**)**

$\dot{x}^i = \dot{x}_0^i + \delta \dot{x}^i$ **(respectively** $\dot{y}^i$**,** $\dot{z}^i$**)**

$x^p = x_0^p + \delta x^p$ **(respectively** y^p**,** $\dot{x}^p$**,**$\dot{y}^p$**,** UT **et** LOD **)**

$T1_k = T1_k^0 + \delta T1_k$ **(respectively** $T2_k$**,...,**$R3_k$**)**

$\dot{T}1_k = \dot{T}1_k^0 + \delta \dot{T}1_k$ **(respectively** $\dot{T}2_k$**,...,**$\dot{R}3_k$**)**

Observation Equation

$$\begin{pmatrix} A1_s & A2_s \end{pmatrix} \begin{pmatrix} \delta\chi_s \\ \\ \delta T_k \end{pmatrix} + B_s = V_s \tag{7}$$

Normal Equation

$$\begin{pmatrix} A1_s^T P_s A1_s & A1_s^T P_s A2_s \\ \\ A2_s^T P_s A1_s & A2_s^T P_s A2_s \end{pmatrix} \begin{pmatrix} \delta\chi_s \\ \\ \delta T_k \end{pmatrix} + \begin{pmatrix} A1_s^T P_s B_s \\ \\ A2_s^T P_s B_s \end{pmatrix} = 0 \tag{8}$$

$\delta\chi_s$ **&** δT_k**: Linearized unknowns**

$A1_s$ **&** $A2_s$ **: Design matrices of partial derivatives**

P_s **: Weights matrix: inverse of variance matrix (**$P_s = \Sigma_s^{-1}$**)**

B_s **: Constant terms vector and** V_s**: Residuals vector**

Design Matrix

Stacking of $\begin{pmatrix} A1^i_s & A2^i_s \end{pmatrix}$**, with** $A^i_s = \begin{pmatrix} 1 & 0 & 0 & 10^{-6}\times x^i_0 & 0 & c\times z^i_0 & -c\times y^i_0 \\ 0 & 1 & 0 & 10^{-6}\times y^i_0 & -c\times z^i_0 & 0 & c\times x^i_0 \\ 0 & 0 & 1 & 10^{-6}\times z^i_0 & c\times y^i_0 & -c\times x^i_0 & 0 \end{pmatrix}$

$$A^p_s = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1000 & 0 \\ 0 & 0 & 0 & 0 & 1000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{f}\times\frac{1000}{15} \end{pmatrix}, \quad A^{\dot{p}}_s = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1000}{y} & 0 \\ 0 & 0 & 0 & 0 & \frac{1000}{y} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{f}\times\frac{1000}{y\times 15} \end{pmatrix}$$

$$A1_s = \begin{pmatrix} . & . & . & . & . & . & . & . & . \\ I & dt^i_s I & 0 & 0 & 0 & \ldots & \ldots & \ldots & \ldots \\ 0 & I & 0 & 0 & 0 & \ldots & \ldots & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & \ldots \\ . & . & . & . & . & . & . & . & . \end{pmatrix}, \quad A2_s = \begin{pmatrix} . & . \\ A^i_s & dt^i_k A^i_s \\ 0 & A^i_s \\ \vdots & \vdots \\ A^p_s & dt^p_k A^{\dot{p}}_s \\ . & . \end{pmatrix}$$

$$\boxed{\begin{aligned} y &= 365.25 \\ f &= 1.002737909350795 \end{aligned}}$$

Normal Matrix

$$\mathcal{N} = \begin{pmatrix} \sum_{s\in S} A1_s^T P_s A1_s & \sum_{s\in s_1} A1_s^T P_s A2_s & \dots & \dots & \sum_{s\in s_K} A1_s^T P_s A2_s \\ \sum_{s\in s_1} A2_s^T P_s A1_s & \sum_{s\in s_1} A2_s^T P_s A2_s & 0 & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \sum_{s\in s_K} A2_s^T P_s A1_s & 0 & \dots & 0 & \sum_{s\in s_K} A2_s^T P_s A2_s \end{pmatrix}$$

and the right hand side vector:

$$\mathcal{K} = \begin{pmatrix} \sum_{s\in S} A1_s^T P_s B_s \\ \sum_{s\in s_1} A2_s^T P_s B_s \\ \vdots \\ \vdots \\ \sum_{s\in s_K} A2_s^T P_s B_s \end{pmatrix}$$

Constraint Handling
TRF implementation by individual techniques (1/2)

The initial NEQ system of space geodesy observations could be written as:

$$N.(\Delta X) = K$$

where $\Delta X = X - X_{apr}$ **(Linearized Unknowns)**

It is a singular system: rank deficiency $=$ nb. of TRF parameters not reduced by the observations.

Lack of information for some TRF parameters:

- **3 (6) origin components (VLBI)**
- **3 (6) orientation components (all techniques)**
- **3 CRF orientation components (VLBI)**

Constraint Handling
TRF implementation by individual techniques (2/2)

Additional constraints are necessary:

- **Tight constraints ($\sigma \geq 10^{-10}$) m**
- **Removable constraints ($\sigma \approx 10^{-5}$) m**
- **Loose constraints($\sigma \geq 1$) m**
- **Minimum constraints (applied over the TRF parameters and not over station coordinates)**

Datum Definition: Minimum Constraints (1/3)

Application of Minimum Constraints (MC) approach based on

theoretical works by many authors, since the 70's on, e.g.:

- **Free Network Adjustment**
- **S-transformation**
- **Minimum/Inner Constraints**

Main Goals:

- **"Best" TRF datum definition**
- **TRF internal consistency: no distortion**
- **Preserve actual quality of space geodesy observations**

Datum Definition : Minimum Constraints (2/3)

The starting point is the standard relation between two TRF's:

$$X_2 = X_1 + A\theta$$

$$\theta = (T1, T2, T3, D, R1, R2, R3, \dot{T}1, \dot{T}2, \dot{T}3, \dot{D}, \dot{R}1, \dot{R}2, \dot{R}3)^T$$

$$A = \begin{pmatrix}
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
1 & 0 & 0 & x_i^0 & 0 & z_i^0 & -y_i^0 & & & & & & & \\
0 & 1 & 0 & y_i^0 & -z_i^0 & 0 & x_i^0 & & & & 0 & & & \\
0 & 0 & 1 & z_i^0 & y_i^0 & -x_i^0 & 0 & & & & & & & \\
 & & & & & & & 1 & 0 & 0 & x_i^0 & 0 & z_i^0 & -y_i^0 \\
 & & & \approx 0 & & & & 0 & 1 & 0 & y_i^0 & -z_i^0 & 0 & x_i^0 \\
 & & & & & & & 0 & 0 & 1 & z_i^0 & y_i^0 & -x_i^0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & .
\end{pmatrix}$$

Datum Definition : Minimum Constraints (3/3)

L.S. yields for θ: $\theta = \overbrace{(A^TA)^{-1}A^T}^{\mathbf{B}}(X_2 - X_1)$

To have X_1 and X_2 be expressed in the same frame (i.e. $\theta = 0$), a "datum definition" equation at Σ_θ level could be written as

$$B(X_2 - X_1) = 0 \qquad (\Sigma_\theta)$$

and in terms of normal equation:

$$B^T\Sigma_\theta^{-1}B(X_2 - X_1) = 0$$

Adding the above equation to the initial NEQ system, we have:

$$(N + B^T\Sigma_\theta^{-1}B)(\Delta X) = K + B^T\Sigma_\theta^{-1}B(X_R - X_{apr})$$

Σ_θ is a diagonal matrix of small variances over the 14 transformation parameters.

Datum Definition in CATREF

The normal equation is singular: rank deficiency of 14 equal to the datum definition parameters. The combined frame could be defined by, e.g.:

- **Fixing to given values 14 parameters among those to be estimated**
- **Using minimum constraint equation over a selected set of stations of a reference TRF solution R. Having**

$$X_R = X_C + A\theta \tag{9}$$

Least squares give $$\theta = \overbrace{(A^TA)^{-1}A^T}^{\mathbf{B}}(X_R - X_C)$$

To have X_C expressed in the same frame as X_R (i.e. $\theta = 0$), we can write

$$B(X_R - X_C) = 0 \qquad (\Sigma_\theta)$$

and in terms of normal equation

$$B^T\Sigma_\theta^{-1}B(X_R - X_C) = 0 \tag{10}$$

Equation (10) is added to the normal equation (8) of the combination model.

Variance Component Estimation

Degree of Freedom Estimator (Unbiased)

$$\sigma_0^2)_s = \frac{v_s^T P_s v_s}{f_s} \tag{11}$$

with v_s the residual vector and f_s the redundancy factor estimated by

$$f_s = 6n_p)_s + n_{eop})_s - \mathrm{tr}(A_s \nu A_s^T P_s) \tag{12}$$

- $n_p)_s$ **: number of points of solution** s
- $n_{eop})_s$ **: number of EOPs of solution** s
- ν **: inverse of the normal matrix**
- A_s **: design matrix of solution** s
- P_s **: inverse of variance matrix of solution** s **(**$P_s = \Sigma_s^{-1}$**)**

Classical Estimator

$$\sigma_0^2)_s = \frac{v_s^T P_s v_s}{6n_p)_s + n_{eop})_s - \frac{6n_p)_s + n_{eop})_s}{n_{obs}} \times n_{unk}} \tag{13}$$

with n_{obs} number of observations and n_{unk} number of unknowns

RMS & Weighted RMS

$$RMS_{3D})_s^2 = \frac{\sum_i^{n_p)_s} v_X^2)_i + v_Y^2)_i + v_Z^2)_i}{f_s^X} \tag{14}$$

with $\boxed{f_s^X = 3n_p)_s - tr_X(A_s \nu A_s^T P_s)}$

$$RMS_{2D})_s^2 = \frac{\sum_i^{n_p)_s} v_E^2)_i + v_N^2)_i}{\frac{2}{3} f_s^X} \tag{15}$$

$$RMS_{Up})_s^2 = \frac{\sum_i^{n_p)_s} v_{Up}^2)_i}{\frac{1}{3} f_s^X} \tag{16}$$

$$\overline{RMS}_{3D})^2 = \frac{\sum_i^{n_p)_s} \frac{v_X^2)_i}{\sigma_X^2)_i} + \frac{v_Y^2)_i}{\sigma_Y^2)_i} + \frac{v_Z^2)_i}{\sigma_Z^2)_i}}{\left(\sum_i^{n_p)_s} \frac{1}{\sigma_X^2)_i} + \frac{1}{\sigma_Y^2)_i} + \frac{1}{\sigma_Z^2)_i}\right) \times \frac{f_s^X}{3n_s}} \tag{17}$$

$$\overline{RMS}_{2D})^2 = \frac{\sum_i^{n_p)_s} \frac{v_E^2)_i}{\sigma_E^2)_i} + \frac{v_N^2)_i}{\sigma_N^2)_i}}{\left(\sum_i^{n_p)_s} \frac{1}{\sigma_E^2)_i} + \frac{1}{\sigma_N^2)_i}\right) \times \frac{2}{3} \times \frac{f_s^X}{3n_s}} \tag{18}$$

$$\overline{RMS}_{Up})^2 = \frac{\sum_i^{n_p)_s} \frac{v_{Up}^2)_i}{\sigma_{Up}^2)_i}}{\left(\sum_i^{n_p)_s} \frac{1}{\sigma_{Up}^2)_i}\right) \times \frac{1}{3} \times \frac{f_s^X}{3n_s}} \tag{19}$$

The same estimators are computed for velocities and EOP parameters.

Other RMS's

$$RMS_{3D})^2 = \frac{\sum_{i}^{n_p)_s} v_X^2)_i + v_Y^2)_i + v_Z^2)_i}{3n_p)_s - 1} \quad (20)$$

$$RMS_{2D})^2 = \frac{\sum_{i}^{n_p)_s} v_E^2)_i + v_N^2)_i}{2n_p)_s - 1} \quad (21)$$

$$RMS_{Up})^2 = \frac{\sum_{i}^{n_p)_s} v_{Up}^2)_i}{n_p)_s - 1} \quad (22)$$

$$\overline{RMS}_{3D})^2 = \frac{\sum_{i}^{n_p)_s} \frac{v_X^2)_i}{\sigma_X^2)_i} + \frac{v_Y^2)_i}{\sigma_Y^2)_i} + \frac{v_Z^2)_i}{\sigma_Z^2)_i}}{\sum_{i}^{n_p)_s} \frac{1}{\sigma_X^2)_i} + \frac{1}{\sigma_Y^2)_i} + \frac{1}{\sigma_Z^2)_i}} \quad (23)$$

$$\overline{RMS}_{2D})^2 = \frac{\sum_{i}^{n_p)_s} \frac{v_E^2)_i}{\sigma_E^2)_i} + \frac{v_N^2)_i}{\sigma_N^2)_i}}{\sum_{i}^{n_p)_s} \frac{1}{\sigma_E^2)_i} + \frac{1}{\sigma_N^2)_i}} \quad (24)$$

$$\overline{RMS}_{Up})^2 = \frac{\sum_{i}^{n_p)_s} \frac{v_{Up}^2)_i}{\sigma_{Up}^2)_i}}{\sum_{i}^{n_p)_s} \frac{1}{\sigma_{Up}^2)_i}} \quad (25)$$

Constraints Handling in CATREF modules

- **Remove Constraints :** $(\Sigma_s^{unc})^{-1} = (\Sigma_s^{est})^{-1} - (\Sigma_s^{const})^{-1}$

- **Add Minimum Constraints (MC):** $(\Sigma_s^{mc})^{-1} = (\Sigma_s^{unc})^{-1} + (B^T \Sigma_\theta^{-1} B)$
  where $B = (A^T A)^{-1} A^T$

- **Compute Solution with MC:** $\overline{X}_s = \Sigma_s^{mc}[(\Sigma_s^{est})^{-1} X_s^{est} - (\Sigma_s^{const})^{-1} X_s^{const}]$

- **Augment the loosely constrained solution by minimum constraints using:**

$$\Sigma_s^{mc} = \Sigma_s^{est} - \Sigma_s^{est} B^T (B \Sigma_s^{est} B^T + \Sigma_\theta)^{-1} B \Sigma_s^{est}$$

Epoch of Minimal Position Variance

For a given station with position vector X at epoch t_s and velocity vector $\dot{X}$, the variance propagation law gives its variance at epoch t as:

$$\begin{aligned} \mathrm{var}(X(t)) &= \mathrm{var}(X(t_s)) + 2 \times (t - t_s) \times \mathrm{cov}(X, \dot{X}) \\ &+ (t - t_s)^2 \times \mathrm{var}(\dot{X}) \end{aligned} \tag{26}$$

The Epoch of Minimal Position Variance t is the epoch which minimizes the variance of the station positions so that:

$$\frac{d[\mathrm{tr}(\mathrm{var}(X(t)))]}{dt} = 0 \tag{27}$$

Note that we minimize the $trace$ of the variance, so that:

$$t = t_s - \frac{[\mathrm{cov}(x, \dot{x}) + \mathrm{cov}(y, \dot{y}) + \mathrm{cov}(z, \dot{z})]}{[\mathrm{var}(\dot{x}) + \mathrm{var}(\dot{y}) + \mathrm{var}(\dot{z})]} \tag{28}$$

CATREF Combination Strategy

- **Remove initial constraints (if any) from individual SINEX files**
- **Apply Minimum Constraints equally to all individual SINEX files**
- **Combine all MC'd SINEX files**
- **Detect and properly handle outliers (de-weight or reject)**
- **Locate discontinuities (in case of time series combination)**
- **Iterate**
- **Scale the variance-covariance matrices by the estimated variance factors**
- **Re-iterate until Variance of Unit Weight (σ_0) is close to unity**