Localisation précise par moyens spatiaux

Astrodynamics: the bare necessities

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Astrodynamics

According to the Encyclopedia of Astronomy and Astrophysics Astrodynamics means *the study of the motion of bodies in gravitational fields, with particular reference to the motions of artificial satellites and space probes.*

Astrodynamics thus deals with the motion of

- Point masses (minor planets, comets, natural and artifical satellites)
- Extended celestial bodies (planets, natural satellites (moons)).

Astrodynamics in the broadest sense also deals with the rotational motion (e.g., of planets).

This lecture is based on the two-volume monograph

G. Beutler: *Methods of Celestial Mechanics:*

Vol 1: Physical, mathematical, and Numerical Principles,

Vol II: Application to Planetary System, geodynamics, and Satellite Geodesy

Springer-Verlag 2005

Astrodynamics

The focus of this lecture uniquely is on the orbital motion of point masses orbiting around the Sun or the Earth as primary gravitational attractors.

The illustrations were

- either generated with *MatLab* or
- or extracted from Beutler (2005).
- or generated with *CelMech*, a menu-driven program system, which allows it, e.g., to numerically integrate the planetary system or to solve the equation of motion of artificial Earth satellites.

CelMech consists of eight eight programs and has a very useful help-system.

Table of Contents

Abbreviations

AM(V)	Angular momentum (vector)
CB(s)	Celestial bodies
CM	Celestial Mechanics
CoM	Center of mass
CS	(Cartesian) Coordinate System
EQ(s)	Equations
DEQ(s)	Differential Equation (system)
IGS	International GNSS Service
IS	Inertial System
JD	Julian Date
LEO	Low Earth Orbiter
LHS	Left-hand side (of an EQ)
MJD	Modified Julian Date
MP	Minor planet

Abbreviations

n-g	non-gravitational
IS	Inertial System
LEO	Low Earth Orbiter
LHS	Left-hand side (of an EQ)
NEQs	Normal EQuation System
RHS	Right-hand side (of an EQ)
t	Newtonian time
$\boldsymbol{x}(t)$	Position vector of a CB in the IS
$\boldsymbol{r}(t)$	heliocentric or geocentric position vector of a CB
RPR	Radiation pressure
SM-axis	Semi-major axis
TT	Terrestrial Time
TAI	Coordinated Atomic time
TBP	Two Body Problem
TDB	Barycentric Dynamic time

Equations of Motion: Planetary System

For our purpose it is sufficient to say that an inertial system (IS) consists of a time scale and a Cartesian coordinate system in the three-dimensional Euclidian space $\mathbf{E}^3$**.** We assume the system to be rigidly attached to the rest system of quasars.

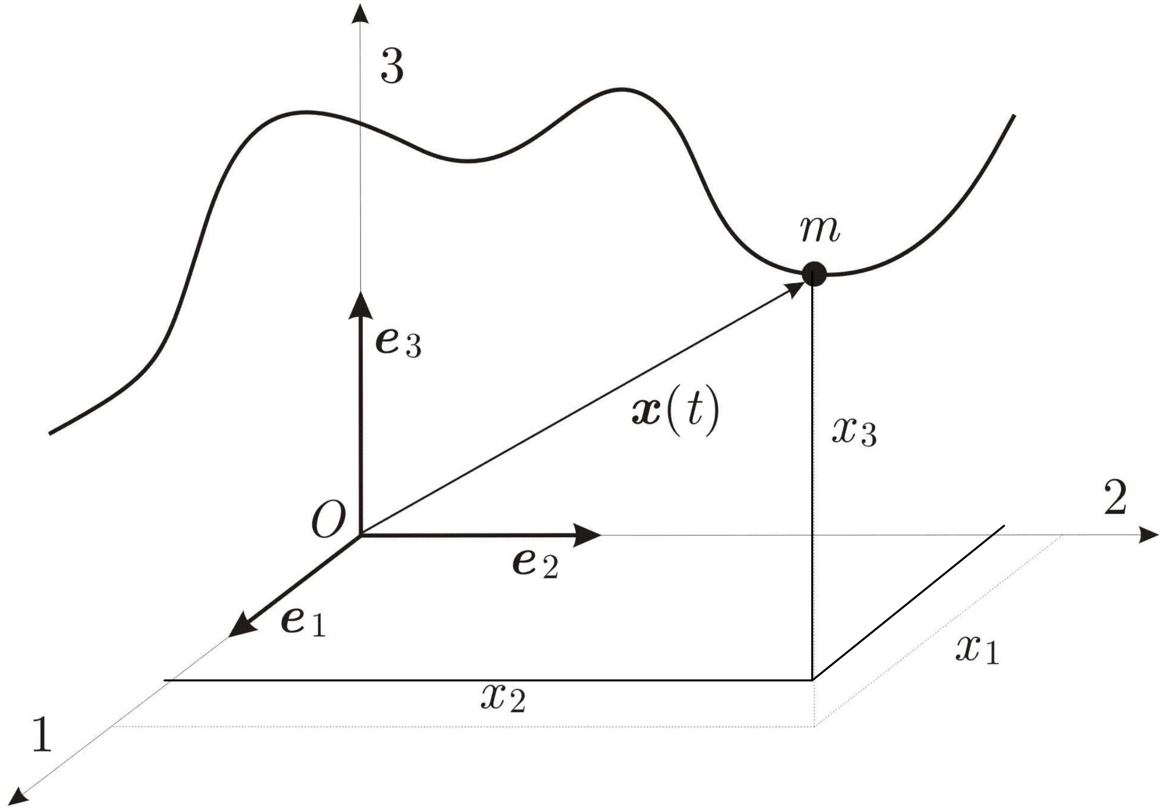
Celestial Mechanics (CM) of the planetary system uses *TDB*, barycentric dynamic time, which is derived from atomic time through:

$$TDB = TT + 0.001658 \sin g + 0.000014 \sin 2g$$

where $g = 357.53 + 0.9856003\ (JD - 2451545)$ and *TT is the* terrestrial time, which in turn follows from *TAI*. We use the ecliptical system *J2000.0*, defined through the mean Earth equator and the mean ecliptic of January 1, 2000, 0^h TBD. In the planetary system we will uniquely take into account Newton's law of universal gravitation specifying the attractive force between point masses *m* and *M at a distance r* (***e*** is the unit vector between *m* and *M*)*:*

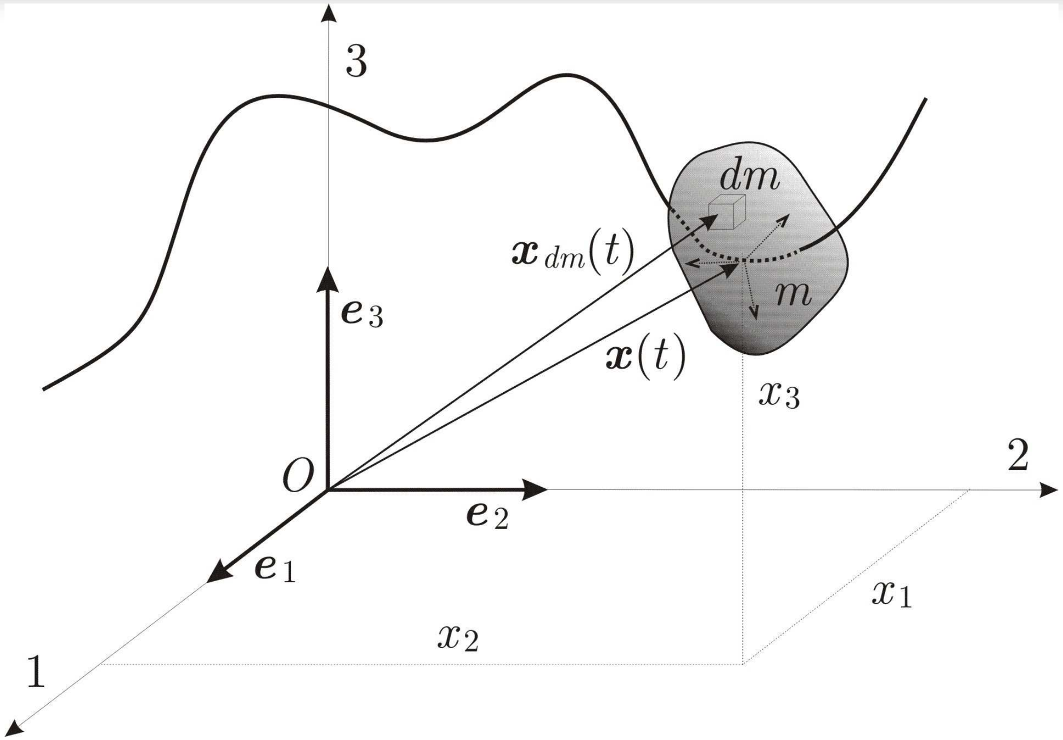
$$\boldsymbol{f} = G\,\frac{m\,M}{r^2}\,\boldsymbol{e}$$

Equations of Motion: Planetary System



The trajectory of the point mass *m* is known, if its position vector ***x****(t)* is known in the IS as a function of time *t.*

Equations of Motion: Planetary System



When dealing with celestial bodies of finite dimensions the position vector ***x***(t) of the body‘s center of mass and its attitude w.r.t. the IS have to be known.

This implies that a body-fixed reference frame has to be known at each epoch *t.*

Equations of Motion: Planetary System

The EQs of motion of the planetary system are those of the *N*-body problem:

$$\frac{d\,(m_i \dot{\boldsymbol{x}}_i)}{dt} = -\,k^2\, m_i \sum_{j=0,j\neq i}^{n} m_j\, \frac{\boldsymbol{x}_i - \boldsymbol{x}_j}{|\boldsymbol{x}_i - \boldsymbol{x}_j|^3}\,, \qquad i = 0, 1, 2, \ldots, n$$

Dividing this EQ by m_i we obtain:

$$\ddot{\boldsymbol{x}}_i = -\,k^2 \sum_{j=0,j\neq i}^{n} m_j\, \frac{\boldsymbol{x}_i - \boldsymbol{x}_j}{|\boldsymbol{x}_i - \boldsymbol{x}_j|^3}\,, \qquad i = 0, 1, 2, \ldots, n$$

Subtracting from the above Eqs for $I \neq 0$ the equation for the central body (Sun) with $i=0$ we obtain the heliocentric equations of motion:

$$\ddot{\boldsymbol{r}}_i = -\,k^2 (m_0 + m_i)\, \frac{\boldsymbol{r}_i}{r_i^3} - k^2 \sum_{j=1,j\neq i}^{n} m_j \left\{ \frac{\boldsymbol{r}_i - \boldsymbol{r}_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|^3} + \frac{\boldsymbol{r}_j}{r_j^3} \right\}$$

The first term on the r.h.s. is the two-body term. The second is the perturbation term. Excluding close encounters $i=1,2,..,n$ and assuming that the masses m_i are significantly smaller than the mass m_0 of the central body, the absolute value of the perturbation term is significantly smaller than that of the two-body term, as well.

Equations of Motion: Planetary System

The planetary system, consisting of the Sun, the planets, the moons of the planets, minor planets, comets, with all in all N=n+1 celestial bodies, is mathematically described by a system of ordinary differential equations (DEQs) of second order and $3 \cdot n$ (scalar) equations.

If the initial state vectors of all CBs (position- and velocity- vectors of all CBs) are known, the DEQs can be solved, e.g., with the methods of numerical analysis.

Minor planets and comets usually are assumed to have negligible masses. Their EQs of motion therefore may be solved separately from the EQs of motion of the CBs with finite masses. The corresponding DEQs, where the m_j und die $\boldsymbol{r}_j$ may be assumed as known, read as:

$$\ddot{\boldsymbol{r}} = -\,k^2\,m_0\,\frac{\boldsymbol{r}}{r^3} \;-\; k^2 \sum_{j=1}^{n} m_j \left\{ \frac{\boldsymbol{r} - \boldsymbol{r}_j}{|\boldsymbol{r} - \boldsymbol{r}_j|^3} + \frac{\boldsymbol{r}_j}{r_j^3} \right\}$$

Equations of Motion: Planetary System

The RHS of the EQs of motion of the planetary system may be written as gradients of (scalar) potentials:

$$\ddot{\boldsymbol{r}}_i = \nabla_i \left\{ U_i + R_i \right\} \ , \quad i = 1, 2, \ldots, n$$

$$U_i = \frac{k^2 \left(m_0 + m_i \right)}{r_i}$$

$$R_i = k^2 \sum_{j=1, j \neq i}^{n} m_j \left\{ \frac{1}{|\boldsymbol{r}_i - \boldsymbol{r}_j|} - \frac{\boldsymbol{r}_i \boldsymbol{r}_j}{r_j^3} \right\}$$

First Integrals: Ten functions of the position and velocity vectors are constant (time independent). Six are due to the fact that the center of mass (CoM) of the planetary system moves with constant velocity on a straight line in each IS. One function exploits the fact that the total energy of the system is constant. Three functions express the preservation of the total angular momentum.

Equations of Motion: Planetary System

Motion of the CoM: The CoM of a system of point masses is defined as:

$$\boldsymbol{X}_0 \stackrel{\text{def}}{=} \frac{1}{M} \sum_{i=0}^{n} m_i \, \boldsymbol{x}_i$$

From the EQs of motion we obtain directly:

$$\frac{1}{M} \sum_{i=0}^{n} m_i \, \ddot{\boldsymbol{x}}_i = \ddot{\boldsymbol{X}}_0 = \boldsymbol{0}$$

Integrating this EQ twice in time we obtain:

$$\boldsymbol{X}_0(t) = \boldsymbol{X}_{00} + \boldsymbol{V}_{00}(t - t_0)$$

where the two vectors on the RHS may be interpreted as the (time-independent position- and velocity-vectors) of the CoM of the planetary system.

Equations of Motion: Planetary System

Preservation of angular momentum (AM): AM is defined by:

$$\boldsymbol{h} \stackrel{\text{def}}{=} \sum_{i=0}^{n} m_i \, \boldsymbol{x}_i \times \dot{\boldsymbol{x}}_i$$

We multiply EQ of motion *i* with m_i, multiply by $\boldsymbol{x_i}$ *x* (cross product) and sum over all CB:

$$\sum_{i=0}^{n} m_i \, \boldsymbol{x}_i \times \ddot{\boldsymbol{x}}_i = -k^2 \sum_{i=0}^{n} \sum_{j=1, j\neq i}^{n} m_i \, m_j \, \frac{\boldsymbol{x}_i \times (\boldsymbol{x}_i - \boldsymbol{x}_j)}{|\boldsymbol{x}_i - \boldsymbol{x}_j|^3}$$

$$\frac{d}{dt}\left[\sum_{i=0}^{n} m_i \, \boldsymbol{x}_i \times \dot{\boldsymbol{x}}_i\right] = k^2 \sum_{i=0}^{n} \sum_{j=0, j\neq i}^{n} m_i \, m_j \, \frac{\boldsymbol{x}_i \times \boldsymbol{x}_j}{|\boldsymbol{x}_i - \boldsymbol{x}_j|^3} = \boldsymbol{0} \; .$$

The preservation of the AM vector follows by integration over time *t.*

Equations of Motion: Planetary System

Preservation of Energy: The energy is defined by:

$$E = T - U \ ,$$

$$T = \tfrac{1}{2} \sum_{i=0}^{n} m_i \ \dot{\boldsymbol{x}}_i^2$$

$$U = k^2 \sum_{i=0}^{n} \sum_{j=i+1}^{n} \frac{m_i \, m_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}$$

Energy preservation follows by multiplication of EQ *i* by $m_i \, \boldsymbol{x}_i$ and by subsequent summation over *i*:

$$\sum_{i=0}^{n} m_i \ \dot{\boldsymbol{x}}_i \cdot \ddot{\boldsymbol{x}}_i \qquad = -\,k^2 \sum_{i=0}^{n} \sum_{j=1, j\neq i}^{n} m_i \, m_j \ \frac{\dot{\boldsymbol{x}}_i \cdot (\boldsymbol{x}_i - \boldsymbol{x}_j)}{|\boldsymbol{x}_i - \boldsymbol{x}_j|^3}$$

$$= -\,k^2 \sum_{j=0}^{n} \sum_{i=0, i\neq j}^{n} m_i \, m_j \ \frac{\dot{\boldsymbol{x}}_j \cdot (\boldsymbol{x}_j - \boldsymbol{x}_i)}{|\boldsymbol{x}_i - \boldsymbol{x}_j|^3}$$

$$= -\,\tfrac{1}{2} \, k^2 \sum_{i=0}^{n} \sum_{j=1, j\neq i}^{n} m_i \, m_j \ \frac{(\dot{\boldsymbol{x}}_i - \dot{\boldsymbol{x}}_j) \cdot (\boldsymbol{x}_i - \boldsymbol{x}_j)}{|\boldsymbol{x}_i - \boldsymbol{x}_j|^3}$$

Equations of Motion: Planetary System

The preservation of the AMV allows it to define a CS well suited for studying the dynamics of the PS. Its fundamental plane is heliocentric and perpendicular to the AMV. (The fundamental plane of a CS is the plane defined by its first two coordinate axes).

The heliocentric plane perpendicular to the AMV is called the

- invariable plane or
- Laplace plane (in honor of Pierre Simon de Laplace 1749-1827).

Its Euler angles w.r.t. ***J2000.0*** are

- $i = 1.58^o$, $\Omega = 107.6^o$

With the exception of the inner- and outermost planets, the inclination angles of the planetary orbits are small w.r.t. the Laplace plane.

Equations of Motion: Planetary System

Table 4.1. Properties of the planetary system

Planet	Axis a [AU]	Ecc. e	Period P [Years]	Mass m^{-1} [$m_\odot^{-1}$]	Incl. i [deg]
Mercury	0.39	0.206	0.24	6023600.00	7.0
Venus	0.72	0.007	0.62	408523.50	3.4
Earth	1.00	0.017	1.00	328900.55	0.0
Mars	1.52	0.093	1.88	3098710.00	1.8
Jupiter	5.20	0.048	11.86	1047.35	1.3
Saturn	9.54	0.056	29.42	3498.00	2.5
Uranus	19.19	0.046	83.75	22960.00	0.8
Neptune	30.06	0.009	163.72	19314.00	1.8
Pluto	39.53	0.249	248.02	130000000.00	17.1

Seven out of eight planets have small eccentricities and inclinations.

The distinction is made between the inner and outer system. IAU's wisdom named Pluto as a dwarf planet. Minor planets are found “adjacent to the boundaries” of the newly defined outer system.

Equations of Motion: Planetary System

If the revolution periods of Jupiter and a MP may be expressed as the quotient of two small integer numbers, the revolution periods are said to be *commensurable*:

$$\frac{P_{♃}}{P} = \frac{n}{n_{♃}} = \frac{k_1}{k_2}$$

Disregarding perturbations the motion of the two CBs is periodic, where the period is called resonance period

$$P_{\text{res}} = k_1\,P = k_2\,P_{♃}$$

The mean synodic period of a planet (from conjunction to conjunction) is obtained as:

$$P_{\text{syn}} = \frac{2\,\pi}{n - n_{♃}} = \frac{P\,P_{♃}}{P_{♃} - P}$$

Resonance and synodic periods are related by:

$$P_{\text{res}} = (k_1 - k_2)\,P_{\text{syn}}$$

Equations of Motion: Artificial Earth Satellites

Preamble:

- The following derivations are in principle applicable to any satellites (with negligible mass) orbiting about any planet or moon of a planet.
- The following formulas, taken from Beutler (2005), do for simplicity always refer to the Earth as central body.
- We are thus deriving the Eqs of motion of an artificial Earth satellite.
- In order to somewhat simplify the theory, we treat the Earth as a *rigid* body of finite dimensions. Through this assumption we avoid the introduction of Tisserand coordinate systems, Beutler (2005), Vol I, Section 3.3.7, p. 91ff.
- The derivation of the EQs of motion of artificial satellites follows the scheme of the derivation laid down in the previous section.

Equations of Motion: Artificial Earth Satellites

Characterization of the problem:

- The mass of the satellite may be neglected w.r.t. mass of the planet (the Earth); it is thus assumed that the artificial Earth satellites do not influence the Earth's orbital and rotational motion.
- Orbits and masses of other CBs are assumed as known.
- The Earth is treated as a CB with finite dimensions. The other CBs involved usually are treated as point masses (with the possible exception of the Moon).
- As opposed to the EQs of motion of the planetary system, non-gravitational forces (actually perturbing accelerations of the satellite due to non-gravitational forces) have to be included as well.
- The satellite mass and its surface properties are of paramount importance for modeling non-gravitational forces.
- Relativistic effects are small and may be treated as perturbations. The key-word is "parameterized Post-Newtonian" equations (see Beutler (2005), Vol I, Sections 3.5 and 4.4).

Equations of Motion: Artificial Earth Satellites

The EQs of motion for a satellite of mass *m* are first set up in an IS. In the non-relativistic approximation we have the change of linear momentum on the LHS of the EQs, the sum of the forces on the RHS:

$$m\,\ddot{\boldsymbol{x}} = -G\,m \int\limits_{V_\oplus} \rho_{p_r}\,\frac{\boldsymbol{x}-\boldsymbol{x}_p}{|\boldsymbol{x}-\boldsymbol{x}_p|^3}\;dV_\oplus$$

$$-\,G\,m \sum_{j=1}^{n} m_j\,\frac{\boldsymbol{x}-\boldsymbol{x}_j}{|\boldsymbol{x}-\boldsymbol{x}_j|^3} \;+\; \sum \boldsymbol{f}_{\mathrm{ng}}$$

The first term on the RHS represents the gravitational attraction by the Earth, where ρ_p is the density at $\boldsymbol{x}_p$ in the Earth's interior.

The second term represents the sum of the gravitational attractions of all CBs included.

The third term stands for the sum of all non-gravitational forces.

Equations of Motion: Artificial Earth Satellites

Dividing both sides of the EQs of motion by the satellite mass *m* we obtain the usual version of the EQs of motion in the IS:

$$\ddot{\boldsymbol{x}} = -GM \int\limits_{V_\oplus} \rho_{p_r} \frac{\boldsymbol{x} - \boldsymbol{x}_p}{|\boldsymbol{x} - \boldsymbol{x}_p|^3} \, dV_\oplus \; - \; G \sum_{j=1}^{n} m_j \frac{\boldsymbol{x} - \boldsymbol{x}_j}{|\boldsymbol{x} - \boldsymbol{x}_j|^3} \; + \; \sum \boldsymbol{a}_{\mathrm{ng}} \; + \; \ldots$$

where G is the gravity constant, *M* the total mass of the Earth (including oceans and atmosphere).

ρ_{pr} is the density at x_p, expressed in units of *M.*

a_{ng} are the non-gravitational forces per mass *m* of the satellite, often somewhat incorrectly expressed as non-g accelerations.

The EQ of motion acting on the Earth's CoM simply is:

$$\ddot{\boldsymbol{x}}_\oplus = -G \sum_{j=1}^{n} m_j \frac{\boldsymbol{x}_\oplus - \boldsymbol{x}_j}{|\boldsymbol{x}_\oplus - \boldsymbol{x}_j|^3}$$

Equations of Motion: Artificial Earth Satellites

Subtracting the EQ of motion for the CoM of the Earth from the EQ of motion of the satellite (both EQs referring to an IS) we obtain the geocentric EQs of motion of the satellite.

The resulting reference system is quasi-inertial: It is for each epoch *t* parallel to the IS, but as it is geocentric, it contains the non-linear motion of Earth's orbital motion:

$$\ddot{\boldsymbol{r}} = -G\,M \int\limits_{V_\oplus} \rho_{p_r} \frac{\boldsymbol{r} - \boldsymbol{r}_p}{|\boldsymbol{r} - \boldsymbol{r}_p|^3}\, dV_\oplus - G \sum_{j=1}^{n} m_j \left\{ \frac{\boldsymbol{r} - \boldsymbol{r}_j}{|\boldsymbol{r} - \boldsymbol{r}_j|^3} + \frac{\boldsymbol{r}_j}{r_j^3} \right\} + \sum \boldsymbol{a}_{\mathrm{ng}}$$

where: $\boldsymbol{r} \stackrel{\mathrm{def}}{=} \boldsymbol{x} - \boldsymbol{x}_\oplus$

In satellite geodesy we use the equatorial system *J2000.0* to express the above equations in coordinates.

Equations of Motion: Artificial Earth Satellites

All gravitational contributions on the RHS of the EQs of motion may be expressed as gradients of a potential:

$$\ddot{\boldsymbol{r}} = GM\,\nabla \int\limits_{V_{\eth}} \frac{\rho_{p_r}}{|\boldsymbol{r} - \boldsymbol{r}_p|}\, dV_{\eth} + G\,\nabla \left[\sum_{j=1}^{n} m_j \left\{ \frac{1}{|\boldsymbol{r} - \boldsymbol{r}_j|} + \frac{\boldsymbol{r}_j \cdot \boldsymbol{r}}{r_j^3} \right\} \right] + \sum \boldsymbol{a}_{\mathrm{ng}}$$

The above integral is evaluated in an Earth-fixed system – alternatively one would have to perform the integration over a time-varying Earth-body, a true nightmare.

Equations of Motion: Artificial Earth Satellites

The EQs of motion were derived as vector equations. We may, however, also understand/interpret them as coordinate EQs in the quasi-inertial equatorial system.

To evaluate the Earth's potential in the Earth-fixed system, the first term has to be transformed back from the Earth-fixed system to the *IS*.

$$\ddot{\boldsymbol{r}} = GM\,\mathbf{T}_{♁}\,\nabla V(\boldsymbol{r}) \; + \; G\,\nabla\left[\sum_{j=1}^{n} m_j \left\{\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_j|} + \frac{\boldsymbol{r}_j \cdot \boldsymbol{r}}{r_j^3}\right\}\right] + \sum \boldsymbol{a}_{\mathrm{ng}}$$

T Is the transformation matrix form the Earth-fixed system to the quasi-inertial system.

The above equation refers to the quasi-inertial geocentric system.

The gradient operation in the first term on the RHS must be performed in the Earth-fixed system.

Equations of Motion: Artificial Earth Satellites

In the Earth-fixed system the potential function of the Earth reads as:

$$V(\boldsymbol{r}) = GM \int\limits_{V_\oplus} \frac{\rho_{p_r}}{|\boldsymbol{r} - \boldsymbol{r}_p|} \, dV_\oplus$$

Using the Laplace equation (which holds in the exterior of the mass distribution), one eventually obtains the following representation (for details we refer to Beutler, Vol. I, Section 3.4.2):

$$V(r,\lambda,\phi) = \frac{GM}{r} \sum_{i=0}^{\infty} \left(\frac{a_\oplus}{r}\right)^i \sum_{k=0}^{i} P_i^k(\sin\phi)\{\, C_{ik}\cos k\lambda + S_{ik}\sin k\lambda \,\}$$

where *r* is the satellite's distance from the Earth's CoM, λ is the satellites spherical longitude, ϕ its spherical latitude.

P..(sin ϕ) are the associated Legendre functions of degree *i* and order *k.*

C_{ik} *and* S_{ik} are the coefficients of the development of the potential into spherical harmonic functions.

Equations of Motion: Artificial Earth Satellites

The coefficients may be expressed as functions of the Earth's density:

$$C_{i0} = \frac{1}{a_{♁}^{i}} \int\limits_{V_{♁}} \rho_{p_r}\, r^i\, P_i(\sin\phi_p)\; dV \;;\quad i \geq 0$$

$$C_{ik} = \frac{2}{a_{♁}^{i}} \frac{(i-k)!}{(i+k)!} \int\limits_{V_{♁}} \rho_{p_r}\, r^i\, P_i^k(\sin\phi_p) \cos k\lambda_p\; dV \;;\quad i,k \geq 0 \;,\; k \leq i$$

$$S_{ik} = \frac{2}{a_{♁}^{i}} \frac{(i-k)!}{(i+k)!} \int\limits_{V_{♁}} \rho_{p_r}\, r^i\, P_i^k(\sin\phi_p) \sin k\lambda_p\; dV \;;\quad i,k > 0 \;,\; k \leq i$$

The coefficients might be calculated, if the density function in the Earth's interior were known.

As this is not the case, the *C..*, *S..* are the parameters of gravity field estimation.

Equations of Motion: Artificial Earth Satellites

The terms of the degrees *i= 0* und *j=1* assume simple values:

$$C_{00} = \int\limits_{V_♁} \rho_{p_r}\, r_p^0 \; dV = 1$$

$$C_{10} = \frac{1}{a_♁} \int\limits_{V_♁} \rho_{p_r}\, r_p \sin\phi_p \; dV = \frac{1}{a_♁} \int\limits_{V_♁} \rho_{p_r}\, r_{p_3} \; dV = \frac{r_3}{a_♁} = 0$$

$$C_{11} = \frac{1}{a_♁} \int\limits_{V_♁} \rho_{p_r}\, r_p \cos\phi_p \cos\lambda_p \; dV = \frac{1}{a_♁} \int\limits_{V_♁} \rho_{p_r}\, r_{p_1} \; dV = \frac{r_1}{a_♁} = 0$$

$$S_{11} = \frac{1}{a_♁} \int\limits_{V_♁} \rho_{p_r}\, r \cos\phi_p \sin\lambda_p \; dV = \frac{1}{a_♁} \int\limits_{V_♁} \rho_{p_r}\, r_{p_2} \; dV = \frac{r_2}{a_♁} = 0\ .$$

provided we adopt the Earth's CoM as the origin of the Earth-fixed coordinate system.

Equations of Motion: Artificial Earth Satellites

The terms of degree *i=2* may be interpreted in a simple way, provided the coordinate axes are selected as the axes of principal inertia:

$$C_{20} = \frac{1}{a_\oplus^2} \int\limits_{V_\oplus} \rho_{p_r} \, r_p^2 \left[\sin^2 \phi_p \; - \; \tfrac{1}{2} \cos^2 \phi_p \right] dV$$

$$= \frac{1}{a_\oplus^2} \int\limits_{V_\oplus} \rho_{p_r} \left[r_{p3}^2 \; - \; \tfrac{1}{2} \left(r_{p1}^2 + r_{p2}^2 \right) \right] dV$$

$$= \frac{1}{M \, a_\oplus^2} \left[\tfrac{1}{2} \left(I_{\oplus \mathcal{F}_{11}} + I_{\oplus \mathcal{F}_{22}} \right) - I_{\oplus \mathcal{F}_{33}} \right] = \frac{1}{M \, a_\oplus^2} \left[\tfrac{1}{2} \left(A_\oplus + B_\oplus \right) - C_\oplus \right]$$

$$C_{21} = \frac{1}{a_\oplus^2} \int\limits_{V_\oplus} \rho_{p_r} \, r_p^2 \, \cos \lambda_p \, \cos \phi_p \, \sin \phi_p \; dV$$

$$= \frac{1}{a_\oplus^2} \int\limits_{V_\oplus} \rho_{p_r} \, r_{p1} \, r_{p3} \; dV = - \frac{1}{M \, a_\oplus^2} \, I_{\oplus \mathcal{F}_{13}} = 0$$

$$S_{21} = \frac{1}{a_\oplus^2} \int\limits_{V_\oplus} \rho_{p_r} \, r_p^2 \, \sin \lambda_p \, \cos \phi_p \, \sin \phi_p \; dV$$

$$= \frac{1}{a_\oplus^2} \int\limits_{V_\oplus} \rho_{p_r} \, r_{p2} \, r_{p3} \; dV = - \frac{1}{M \, a_\oplus^2} \, I_{\oplus \mathcal{F}_{23}} = 0$$

C_{20} represents the flattening of the Earth. *A.., B.., and C..* are the Earth's three principal moments of inertia.

Equations of Motion: Artificial Earth Satellites

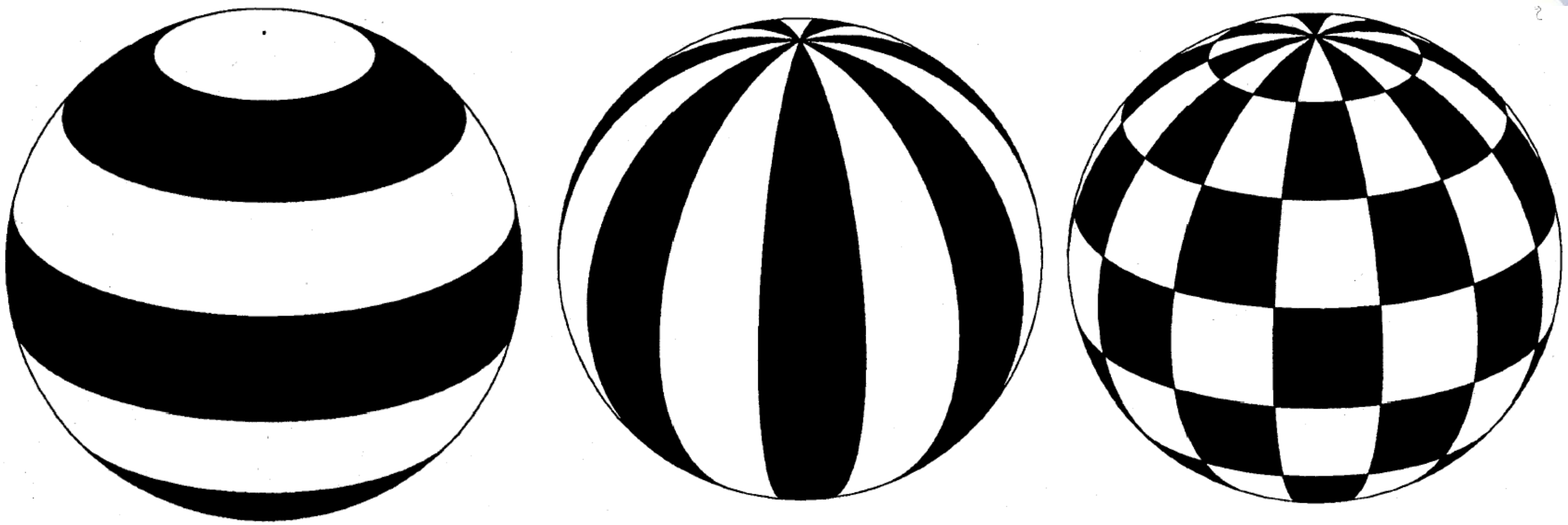
The development of the Earth's potential into spherical harmonics:

- Each solution of the Laplace-EQ is called a harmonic function.
- When expressed in spherical coordinates, one also speaks of spherical-harmonic functions, or simply of spherical harmonics.
- The potential function of the Earth has terms of the following kind:
  - *Zonal terms*, not depending on λ (but only of r und ϕ),
  - *sectorial terms*, not*) depending on ϕ (but only on λ and r), as well as
  - *Tesseral terms*, depending on all three spherical coordinates.

*) Apart from the weighting factor $|\mathbf{cos}\ \phi|^k$ for terms of degree and order k.

The following pictures represent the three different kinds of terms – on a unit sphere. Zones where the potential function has a positive sign are marked in white, those with a negative sign in black.

Equations of Motion: Artificial Earth Satellites



From left to right: Zonal ***(6,0)***, sectorial ***(7,7)*** and tesseral harmonic ***(13,7)*** functions.

Equations of Motion: Artificial Earth Satellites

Using fully normalized Legendre functions, the corresponding harmonics are normalized for $r=a$. and the coefficients C_{ik}, S_{ik} are a measure for the corresponding terms.

The first few coefficients of the potential are (model JGM3):

Coefficient	Value	Coefficient	Value
GM	$398.60044150 \cdot 10^{12}\ \mathrm{m^3s^{-2}}$	$a_{♁}$	$6378136.30\ \mathrm{m}$
$\bar{C}_{20}$	$-0.48416954845647 \cdot 10^{-3}$	$\bar{S}_{20}$	
$\bar{C}_{21}$	$-0.18698764000000 \cdot 10^{-9}$	$\bar{S}_{21}$	$+0.11952801000000 \cdot 10^{-8}$
$\bar{C}_{22}$	$0.24392607486563 \cdot 10^{-5}$	$\bar{S}_{22}$	$-0.14002663975880 \cdot 10^{-5}$
$\bar{C}_{30}$	$+0.95717059088800 \cdot 10^{-6}$	$\bar{S}_{30}$	
$\bar{C}_{31}$	$+0.20301372055530 \cdot 10^{-5}$	$\bar{S}_{31}$	$+0.24813079825561 \cdot 10^{-6}$
$\bar{C}_{32}$	$+0.90470634127291 \cdot 10^{-6}$	$\bar{S}_{32}$	$-0.61892284647849 \cdot 10^{-6}$
$\bar{C}_{33}$	$+0.72114493982309 \cdot 10^{-6}$	$\bar{S}_{33}$	$+0.14142039847354 \cdot 10^{-5}$
$\bar{C}_{40}$	$+0.53977706835730 \cdot 10^{-6}$	$\bar{S}_{40}$	
$\bar{C}_{41}$	$-0.53624355429851 \cdot 10^{-6}$	$\bar{S}_{41}$	$-0.47377237061597 \cdot 10^{-6}$
$\bar{C}_{42}$	$+0.35067015645938 \cdot 10^{-6}$	$\bar{S}_{42}$	$+0.66257134594268 \cdot 10^{-6}$
$\bar{C}_{43}$	$+0.99086890577441 \cdot 10^{-6}$	$\bar{S}_{43}$	$-0.20098735484731 \cdot 10^{-6}$
$\bar{C}_{44}$	$-0.18848136742527 \cdot 10^{-6}$	$\bar{S}_{44}$	$+0.30884803690355 \cdot 10^{-6}$

Equations of Motion: Artificial Earth Satellites

We know that $C_{00}=1$. We may thus summarize:

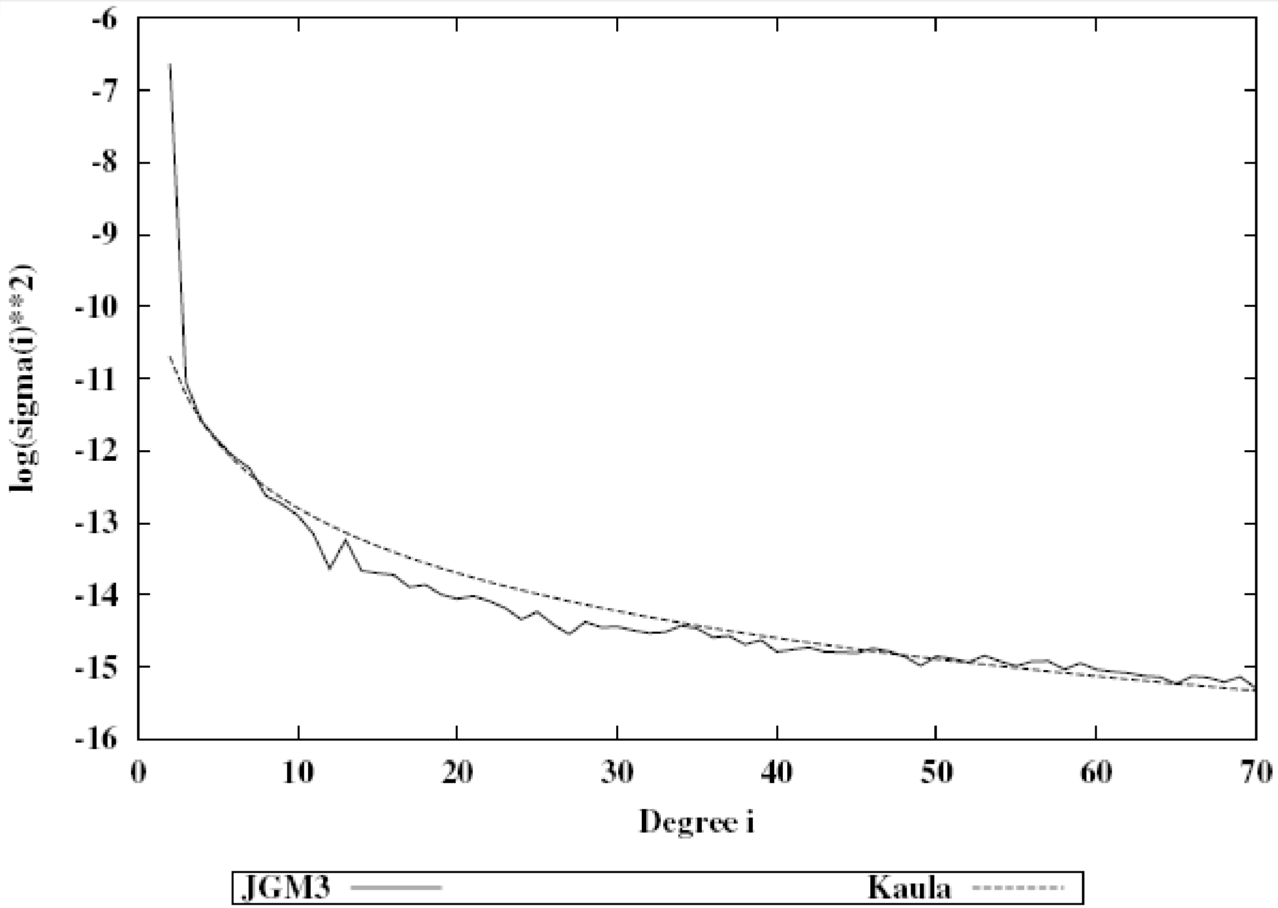
- The term C_{20} is small compared to the main term (factor of 2000).
- The term C_{20} is the largest perturbation terms, by three orders of magnitude larger than all the others. The term is caused by the difference of the equatorial and polar moments of inertia (A & B and C, respectively).
- The other perturbation terms are of a similar order of magnitude, but about 200-1000 times smaller than C_{20} .

We may thus conclude that the that an accurate description of the Earth potential becomes relatively complicated for degrees $n > 2$-

Kaula's rule of thumb gives an approximate "law" for the order of magnitude of the fully normalized geopotential terms as a function of the degree i:

$$\sigma_i^2 \stackrel{\text{def}}{=} \sum_{k=0}^{i} \left[\bar{C}_{ik}^2 + \bar{S}_{ik}^2 \right] \qquad \sigma_i^2 = \frac{160 \cdot 10^{-12}}{i^3}$$

Equations of Motion: Artificial Earth Satellites



Kaula‘s „rule of thumb“ and the „true“ degree amplitudes of degree *i* of the *JGM3.*

Orbit Parameterization

The EQs of motion of an artificial Earth satellite moving under the gravitational attraction of Earth, Moon, Sun, planets and of non-gravitational forces may be written as:

$$\ddot{\mathbf{r}} = -GM\frac{\mathbf{r}}{r^3} + \delta\mathbf{f}(t, \mathbf{r}, \dot{\mathbf{r}})$$

If the state vector ($\boldsymbol{r}(t_0)$, $\boldsymbol{v}(t_0)$) is known at an initial epoch t_0, a particular solution of the above DEQ is defined.

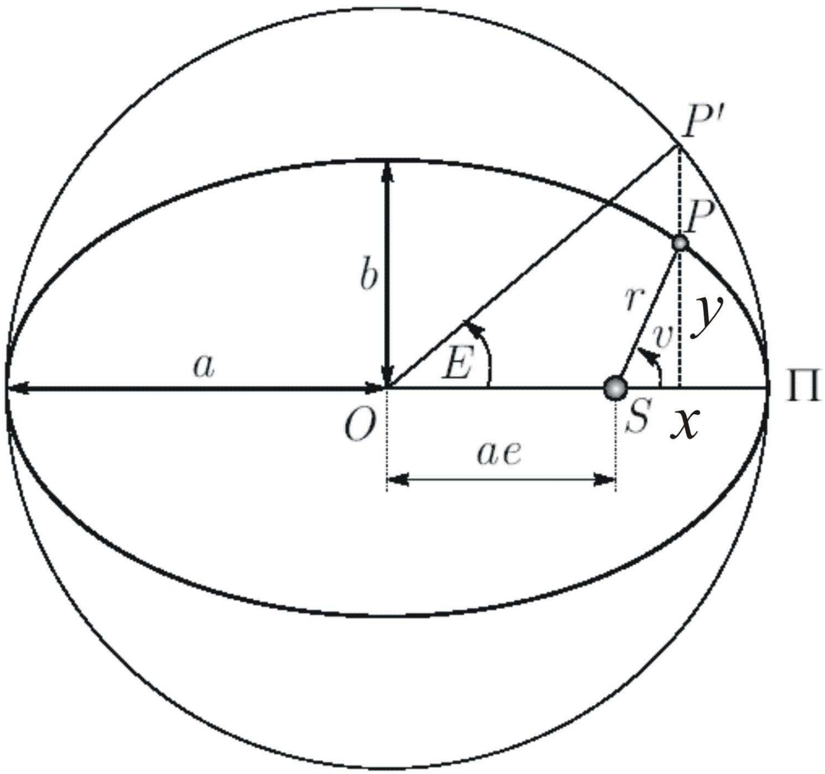
There is a one-to-one relationship between the state vector and the so-called osculating orbital elements of an epoch *t:*

$$r(t), \dot{r}(t) \Leftrightarrow a(t), e(t), i(t), \Omega(t), \omega(t), T_0(t)$$

where *a* stands for the semi-major axis, *e* for the numerical eccentricity, *i* for the inclination w.r.t. the equator, Ω for the right ascension of the ascending node, ω for the distance of the preigee from the node, and T_0 for perigee passing time.

Orbital Coordinate System (x,y,z(=0))

$\mathbf{e}_x$-axis: geocentric direction to perigee, z-axis: $\mathbf{e}_z = (\mathbf{r} \times \mathbf{v} / |\mathbf{r} \times \mathbf{v}|)$, y-axis: $\mathbf{e}_y = \mathbf{e}_z \times \mathbf{e}_x$



v is the so-called true anomaly

Coordinates in orbital system:

n=mean motion (rad/s)

$$n^2 a^3 = GM \qquad (4.32b)$$

$\sigma(t) = n\,(t - T_0)$ (=mean anomaly)

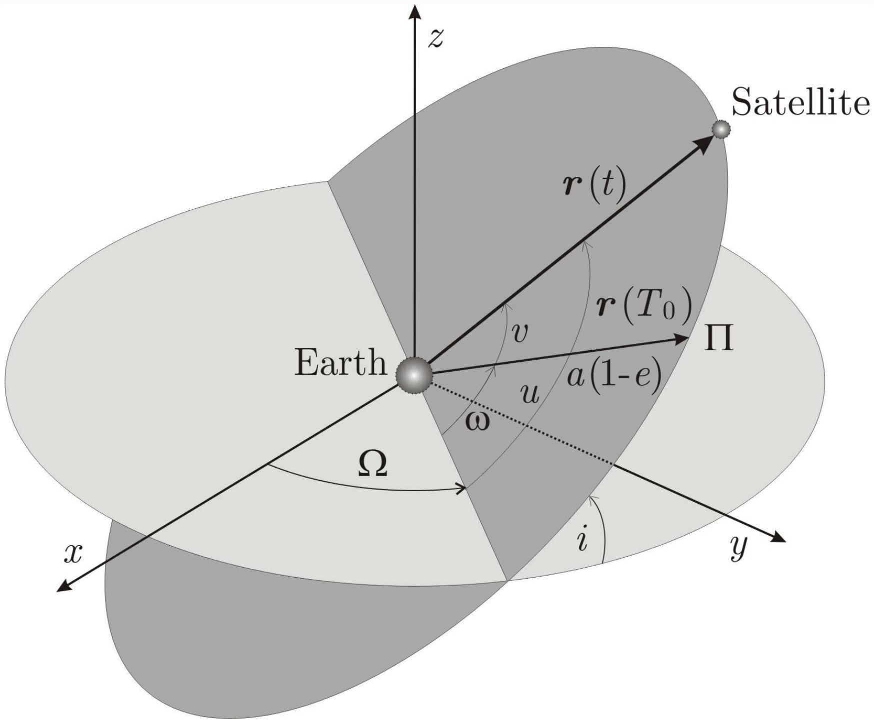
Kepler's equation:

$$E(t) = \sigma(t) + e \sin E(t) \qquad (4.32c)$$

$E(t)$ = eccentric anomaly

$$x = a(\cos E - e)$$
$$y = a\sqrt{1-e^2}\,\sin E \qquad (4.32d)$$

Orbit Transformation Orbital → IS



x and y are the coordinates in the orbital system.

x_a, y_a, z_a the coordinates in the IS (equatorial system *J2000.0*).

$\mathbf{R}_i(\alpha)$ is the rotation matrix about axis *i* and around the angle α.

The transformation from the orbital system to the IS then reads as (dotted quantities stand for the time derivatives):

$$\begin{pmatrix} x_a \\ y_a \\ z_a \end{pmatrix} = \mathbf{R}_3(-\Omega)\cdot\mathbf{R}_1(-i)\cdot\mathbf{R}_3(-\omega)\cdot\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \dot{x}_a \\ \dot{y}_a \\ \dot{z}_a \end{pmatrix} = \mathbf{R}_3(-\Omega)\cdot\mathbf{R}_1(-i)\cdot\mathbf{R}_3(-\omega)\cdot\begin{pmatrix} \dot{x} \\ \dot{y} \\ 0 \end{pmatrix}$$

Orbital Coordinate Systems

Table 4.3. Orbital coordinate systems

System	First unit vector	Transformation from Inertial System $\mathcal{I}$	
Ω	$\boldsymbol{e}_\Omega = \frac{\boldsymbol{e}_3 \times \boldsymbol{h}}{h}$	$\boldsymbol{r}_\Omega =$	$\mathbf{R}_1(i)\,\mathbf{R}_3(\Omega)\,\boldsymbol{r}$
Π	$\boldsymbol{e}_\Pi = \frac{\boldsymbol{q}}{q}$	$\boldsymbol{r}_\Pi =$	$\mathbf{R}_3(\omega)\,\mathbf{R}_1(i)\,\mathbf{R}_3(\Omega)\,\boldsymbol{r}$
$\mathcal{R}$	$\boldsymbol{e}_\mathcal{R} = \frac{\boldsymbol{r}}{r}$	$\boldsymbol{r}_\mathcal{R} =$	$\mathbf{R}_3(u)\,\mathbf{R}_1(i)\,\mathbf{R}_3(\Omega)\,\boldsymbol{r}$
$\mathcal{T}$	$\boldsymbol{e}_\mathcal{T} = \frac{\dot{\boldsymbol{r}}}{\lvert\dot{\boldsymbol{r}}\rvert}$	$\boldsymbol{r}_\mathcal{T} =$	$\mathbf{R}_3(\xi)\,\mathbf{R}_3(\omega)\,\mathbf{R}_1(i)\,\mathbf{R}_3(\Omega)\,\boldsymbol{r}$

Beutler (2005), Vol. 1, Table 4.3 makes the distinction of four orbital systems.
The one we defined previously is the second one of the above table.
In perturbation theory we will need the third or the fourth of these systems.

Elements of perturbation theory: Contents

1. Osculating and mean orbital elements
2. Motivation and classification
3. Gaussian perturbation equations
4. Perturbations of the first order

Osculating and mean elements

The two-body problem allows it to calculate for each epoch *t* the state vector (***r***(*t*), ***v***(*t*)) using a set of orbital elements (a, e, i, Ω, ω, T_0).

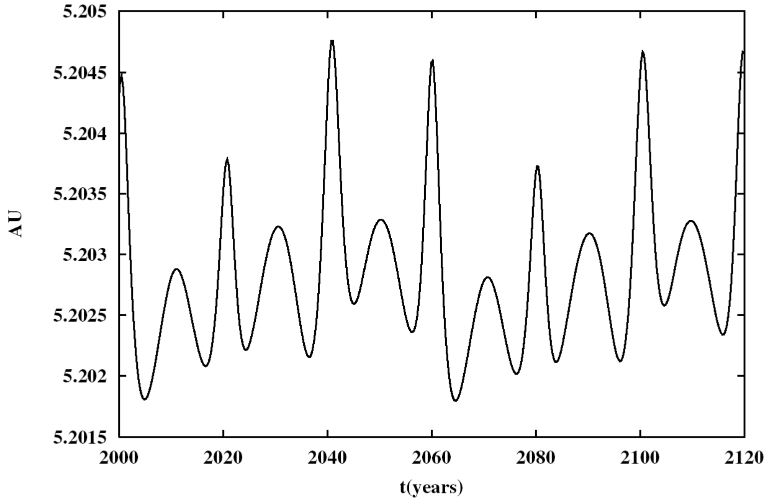
Vice-versa this (time-independent) set of elements may be calculated from the state vector of each epoch *t* :

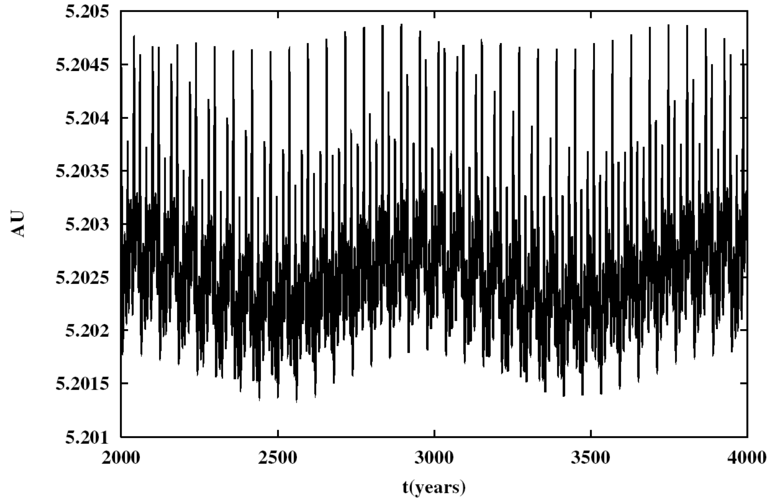
$$t : \{\boldsymbol{r}(t), \dot{\boldsymbol{r}}(t)\} \leftrightarrow \{a, e, i, \Omega, \omega, T_0\}$$

For a perturbed trajectory solving the DEQ on page 34 one may assign a set of osculating element to each epoch *t* by calculating orbital elements from the corresponding state vector using the formulas of the two-body problem. Such elements are called *osculating orbital elements* (at osculation epoch *t*):

$$t : \{\boldsymbol{r}(t), \dot{\boldsymbol{r}}(t)\} \rightarrow \{a(t), e(t), i(t), \Omega(t), \omega(t), T_0(t)\}$$

Osculating and mean elements





Osculating elements are unsuitable to describe the development of an orbit over long time intervals (hundreds of revolutions). The above Figure (left) shows the development of Jupiter's semi-major axis over 120 years (when integrating the outer planetary system). The figure shows a prominent periods of about 60 years (= five revolutions of Jupter, 2 of Saturn).

The above Figure (right), showing the development of Jupiter's osculating semi-major axis years over 2000 years, motivates the definition of *mean orbital elements.*

Osculating and mean elements

Let $I(t) \in \{ a(t), e(t), i(t), \Omega(t), \omega(t), T_0(t) \}$ one of the osculating orbital elements. A mean orbital element at epoch *t* is defined by:

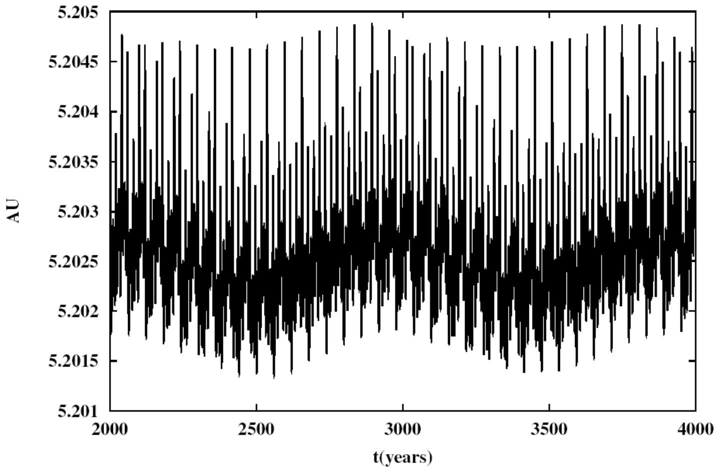
$$\bar{I}(t;\Delta t(t)) = \frac{1}{\Delta t} \int_{t-\Delta t/2}^{t+\Delta t/2} I(t')\ dt'$$

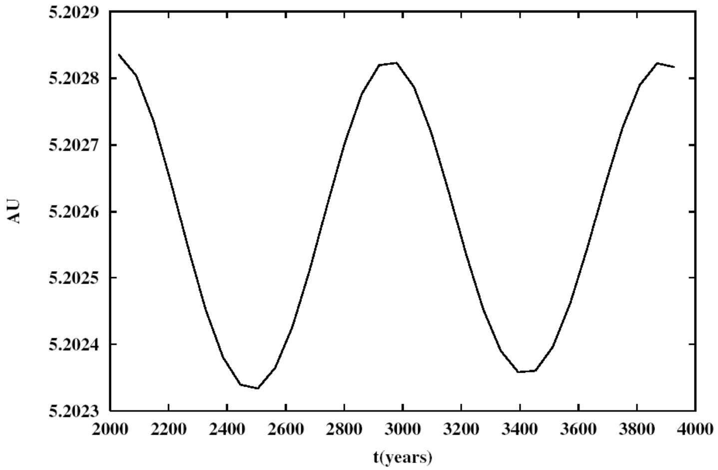
By construction the mean elements are analytical functions of time *t*. The averaging period Δt should either be very long or a multiple of the prevailing short periods.

The following example meets this requirement..

Unskillfully selected Δt may result in the generation of alias periods.

Osculating and mean elements





Osculating (left) und mean (right) semi-major axis *a* of Jupiter over 2000 years. Δt = *5 revs* of Jupiter = *2 revs* of Saturn.

The period of 900 is called the great inequality. It was detected in the early 17th century (Kepler era) as excursions in the longitudes of the planets Jupiter and Saturn, *not* in the semi-major axes.

The effect was correctly explained by Pierre-Simon de Laplace in 1787.

Perturbations: Classification

The expression *perturbed motion* implies that an *unperturbed motion exists*. In CM the unperturbed motion is the two-body motion.

The perturbed motion is defined by the EQs of motion (in this context called *perturbation equation*) and the associated state vector at t_0:

$$\ddot{\boldsymbol{r}} = -\mu \frac{\boldsymbol{r}}{r^3} + \delta \boldsymbol{f}(t, \boldsymbol{r}, \dot{\boldsymbol{r}}) \ ,$$

$$\boldsymbol{r}(t_0) = \boldsymbol{r}_0 \quad \text{and} \quad \dot{\boldsymbol{r}}(t_0) = \boldsymbol{v}_0$$

The first term on the RHS of the above DEQ may be called *two-body term*, the second *perturbation term*. The terminology makes sense, provided:

$$|\ \delta \boldsymbol{f}\ | \ll \left| -\mu \frac{\boldsymbol{r}}{r^3} \right|$$

Perturbations: Classification

All methods to solve the perturbation equations are called

- Methods of perturbation theory or simply perturbation method.

In CM we make the distiction between

- General perturbation methods, if the solution is sought in terms of elementary inegrable functions
- Special perturbation methods, if the solution is generated numerically.

General perturbation theory played a preeminent role in CM and in the context of the development of analytical mechanics..

With the advent of electronic computers in the 2nd half of the 20th century special perturbation methods replaced the general methods for most applications.

General perturbation methods are important today for studies trying to understand particular perturbed motions.

Gaussian Perturbation Equations

Let $I(t) \in \{ a(t), e(t), i(t), \Omega(t), \omega(t), T_0(t) \}$ an osculating element.

By construction their time dependence is given by that of the state vector:

$$I(t) \stackrel{\text{def}}{=} I\left(\boldsymbol{r}(t), \dot{\boldsymbol{r}}(t)\right)$$

The time development of *I(t)* is obtained by taking the the time derivative of the above EQ:

$$\dot{I} = \sum_{l=1}^{3} \left\{ \frac{\partial I}{\partial r_l}\,\dot{r}_l + \frac{\partial I}{\partial \dot{r}_l}\,\ddot{r}_l \right\} = \nabla_r I \cdot \dot{\boldsymbol{r}} + \nabla_v I \cdot \ddot{\boldsymbol{r}}$$

The second time derivative of ***r*** in the second term on the RHS may be replaced by using the perturbation EQ in rectangular coordinates:

$$\dot{I} = \nabla_r I \cdot \dot{\boldsymbol{r}} + \nabla_v I \cdot \left\{ -\mu\,\frac{\boldsymbol{r}}{r^3} + \delta \boldsymbol{f} \right\}$$

Gaussian Perturbation Equations

As *I(t)=const* in the two-body motion, we obtain the remarkably simple Gaussian perturbation equations:

$$\dot{I} = \nabla_v I \cdot \delta \boldsymbol{f}$$

The six perturbation equations in the osculating elements may thus be written as a systems of 6 coupled DEQs of order 1:

$$\dot{I}_k = \nabla_v I_k \cdot \delta \boldsymbol{f} \ , \quad k = 1, 2, \ldots, 6$$

The DEQ system is mathematically equivalent to the perturbation EQs in rectangular coordinates (3 DEQs of second order).

The alternative formulation is attractive, as the RHS are small quantities.

To obtain the Gaussian perturbation equations in explicit form for a particular perturbation $\delta\boldsymbol{f}$, one merely has to calculate the scalar products on the RHS.

Gaussian Perturbation Equations

Let us directly derive the perturbation equation for *a* from energy preservation of the two-body motion:

$$\frac{\mu}{a} = \frac{2\,\mu}{r} - \dot{\boldsymbol{r}}^2$$

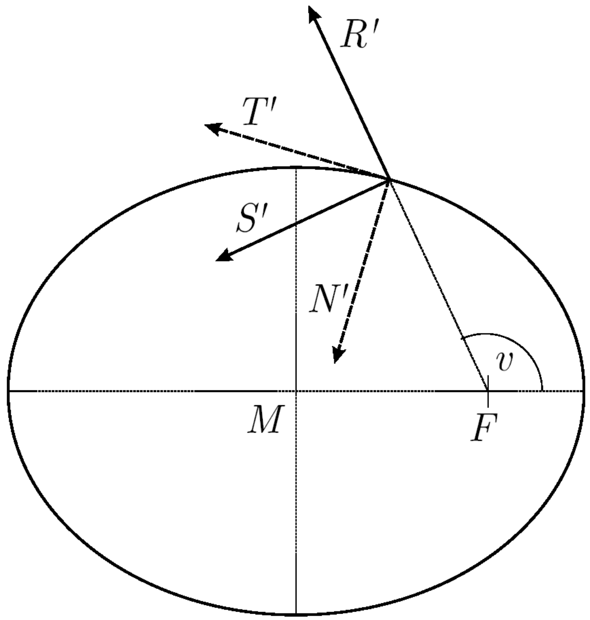
Taking the gradient (in velocity space) on both sides of the Eq we obtain:

$$\nabla_v \left(\frac{\mu}{a}\right) = -\,\frac{\mu}{a^2}\,\nabla_v a = -\,2\,\dot{\boldsymbol{r}} \qquad \text{or:} \qquad \nabla_v a = \frac{2\,a^2}{\mu}\,\dot{\boldsymbol{r}}$$

The Gaussian perturbation EQ for the semi-major axis *a thus* reads as:

$$\dot{a} = \frac{2\,a^2}{\mu}\,\dot{\boldsymbol{r}} \cdot \delta \boldsymbol{f}$$

Gaussian Perturbation Equations



Scalar products are invariant under rotation. Therefore one may use particularly well suited decompositions of the forces.

The figure illustrates the *(R,S,W)*- and die *(T,N,W)*-decomposition*.* *R* stands for the radial, *S for* normal to *R* in the orbital plane (close to along-track). *W* is parallel to $\boldsymbol{e}_R \times \boldsymbol{e}_S$, *i.e.,* normal to the orbital plane.

T is the along-track component, *N is* normal to *T* pointing into the interior of the ellipse*.*

Many perturbations may be particularly simply represented in one of the two decompositions (e.g., drag and radiation pressure).

Gaussian Perturbation Equations

Using the *(R,S,W)*-decomposition the Gaussian perturbation equations for the six classical osculating elements read as:

$$
\begin{aligned}
\dot{a} &= \sqrt{\frac{p}{\mu}}\,\frac{2\,a}{1-e^2}\left\{e\,\sin v\,R' + \frac{p}{r}\,S'\right\} \\
\dot{e} &= \sqrt{\frac{p}{\mu}}\left\{\sin v\,R' + (\cos v + \cos E)\,S'\right\} \\
\dot{T}_0 &= -\frac{1-e^2}{n^2\,a\,e}\left\{\left(\cos v - 2\,e\,\frac{r}{p}\right)R' - \left(1+\frac{r}{p}\right)\sin v\,S'\right\} - \frac{3}{2\,a}\,(t-T_0)\,\dot{a} \\
\frac{di}{dt} &= \frac{r\,\cos u}{n\,a^2\,\sqrt{1-e^2}}\,W' \\
\dot{\Omega} &= \frac{r\,\sin u}{n\,a^2\,\sqrt{1-e^2}\,\sin i}\,W' \\
\dot{\omega} &= \frac{1}{e}\sqrt{\frac{p}{\mu}}\left\{-\cos v\,R' + \left(1+\frac{r}{p}\right)\sin v\,S'\right\} - \cos i\,\dot{\Omega}\ ,
\end{aligned}
$$

The Gaussian perturbation equations hold for all types of forces (conservative or non-conservative).

Gaussian Perturbation Equations

The Gaussian perturbation equations may be solved approximately making use of the fact that the RHS of the Gaussian perturbation EQs are small quantities.

Perturbation methods of first order result, if the RHS of the EQs are approximated with the two-body approximation (keeping the osculating elements constant).

In this case the coupled system of six first-order DEQs becomes decomposed into six mutually independent integrals, which may be solved independently.

The scalar products on the RHS of the perturbation equations are thus calculated with the two-body approximation.

Perturbation methods of the first order are very efficient. Note, however, that the results are approximate and should not be used over more than a few revolutions.

Perturbations due to C_{20}

Let us assume that:

- only the terms C_{00} and C_{20} are different from zero
- the Earth's rotation axis coincides with the Earth's axis of maximum inertia
- Precession, nutation, and polar motion are neglected.

Under these assumptions we may write:

$$V(r,\phi) = \frac{GM}{r} + GM\, a_{\oplus}^2\, C_{20}\, \frac{1}{r^3} \left(\frac{3}{2} \sin^2 \phi - \frac{1}{2} \right)$$

Note that

- the Earth's potential is longitude-independent
- the transformation between the IS and the Earth-fixed system may be set to the unit matrix.

Perturbations due to C_{20}

The perturbations were provided in the equatorial system. They may be easily transformed into the (*R,S,W*)-system:

$$\begin{pmatrix} R' \\ S' \\ W' \end{pmatrix} = \mathbf{R}_3(u)\ \mathbf{R}_1(i)\ \mathbf{R}_3(\Omega)\ \frac{3}{2}\frac{\tilde{C}_{20}}{r^5} \begin{pmatrix} r_1\left(1 - 5\,\frac{r_3^2}{r^2}\right) \\ r_2\left(1 - 5\,\frac{r_3^2}{r^2}\right) \\ r_3\left(3 - 5\,\frac{r_3^2}{r^2}\right) \end{pmatrix}$$

Putting $u=\omega+v$ the result is (see Beutler (2005), Vol. 2, Sect 3.1.2):

$$\begin{pmatrix} R' \\ S' \\ W' \end{pmatrix} = \frac{3}{2}\ \frac{\tilde{C}_{20}}{r^4} \begin{pmatrix} 1 - \frac{3}{2}\sin^2 i + \frac{3}{2}\sin^2 i\ \cos 2u \\ \sin^2 i\ \sin 2u \\ \sin 2i\ \sin u \end{pmatrix}$$

Perturbations due to C_{20}

The previous expressions may be introduced into the RHS of the Gaussian perturbation EQs .

Assuming the orbital elements on the RHS of these EQs as constant, the EQs may be solved in an elementary way.

Assume circular motion, adapt the Gaussian perturbation EQs to this case and solve them!

$$\dot{a} = \sqrt{\frac{p}{\mu}} \frac{2a}{1-e^2} \left\{ e \sin v \, R' + \frac{p}{r} S' \right\}$$

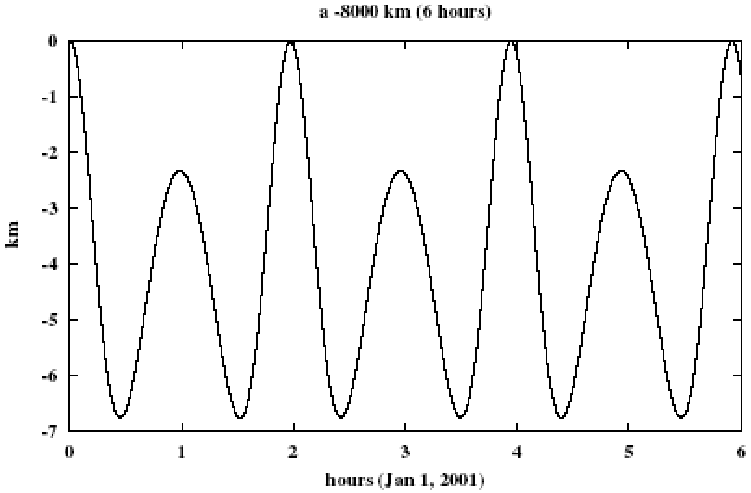
$$\dot{\Omega} = \frac{r \sin u}{n a^2 \sqrt{1-e^2} \sin i} W'$$

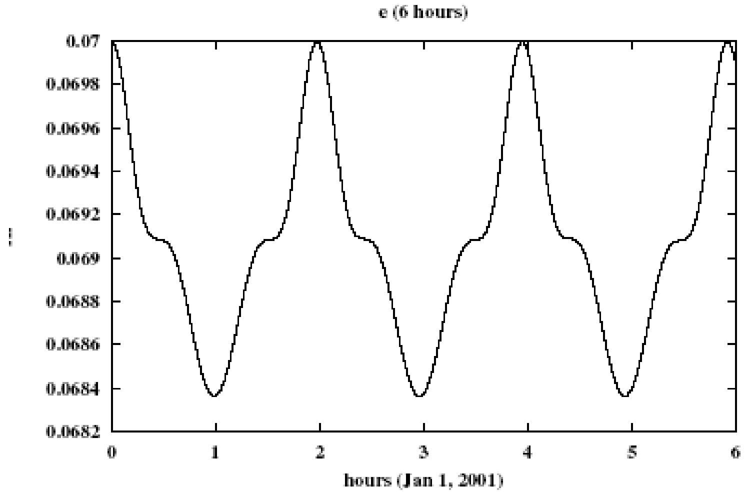
Perturbations due to C_{20}

Subsequently the perturbation EQs will be solve without approximat-ions and the resulting osculating elements will be discussed. The osculating elements at t_0 = *Jan 1, 2001, 0^h* are:

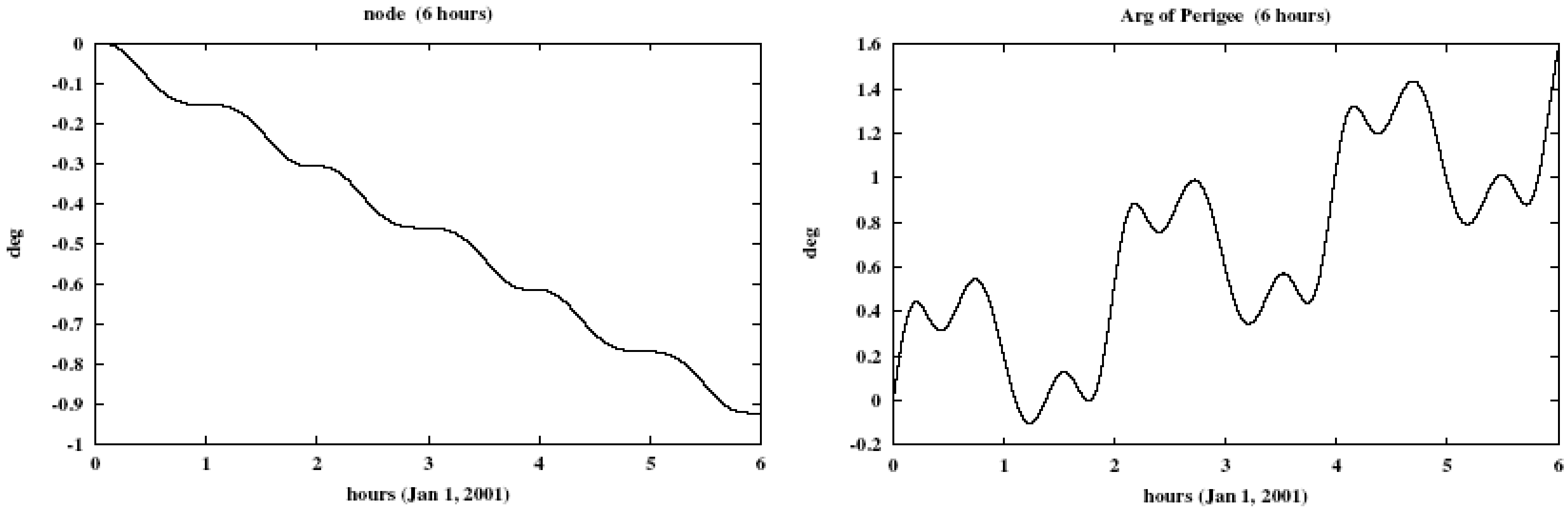
Element	Value	Element	Value
a	8000 km	e	0.07
i	$35°$	Ω	$0°$
ω	$0°$	T_0	t_0

a and *e* show only short-period variations.



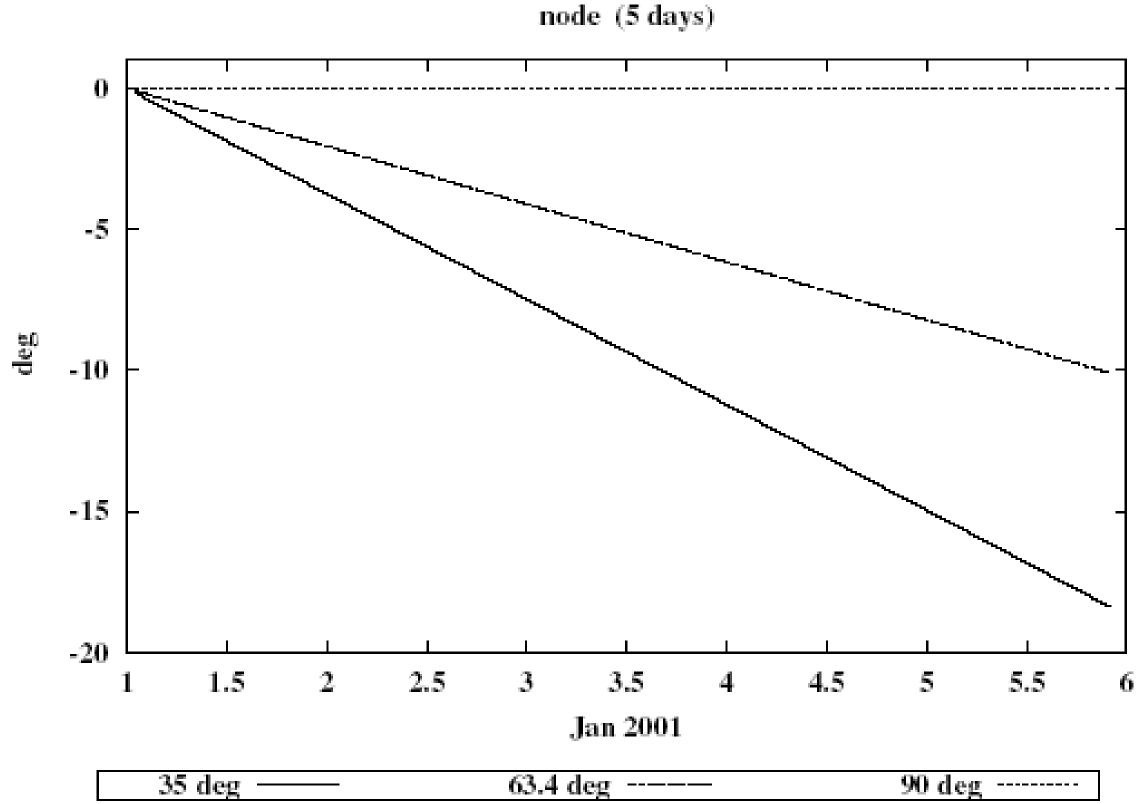


Perturbations due to C_{20}



The perturbations in the R.A. of the ascending node Ω and in the argument ω of perigee also show secular perturbations (the node rotates clockwise, the perigee ant-clockwise – for the initial conditions used).

Perturbations due to C_{20}



Rotation of the node for $i = 35^o, 63.4^o$ and 90^o. The node does not rotate for $i = 90^o$ (and $i = 0^o$). For $i > 90^o$, the node progresses anti-clockwise.

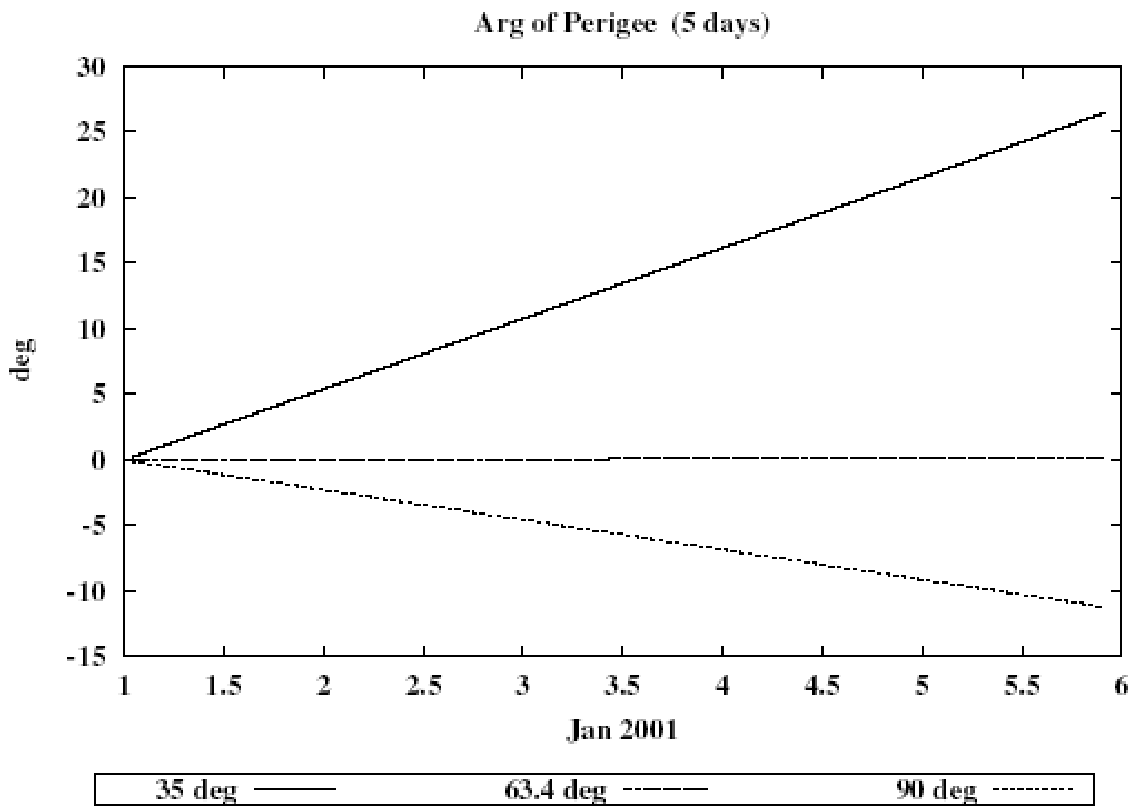
Perturbations due to C_{20}

Height h [km]	Inclination i [°]	Revolution Period U [min]
400	97.03	92.6
600	97.79	96.7
800	98.61	100.9
1000	99.48	105.1
1200	100.42	109.4

Sun-synchronous orbits for different sm-axes (heights): The orbital plane must rotate anti-clockwise with *1°* / day, to keep the orientation of the orbital plane (approximately) constant w.r.t. the Sun. The above table gives the inclinations as a function of the orbit height, for which this the case.

The anti-clockwise motion of the node for *i>0°* may thus be used for LEOs to generate Sun-synchronous orbits (the GOCE orbit, e.g., lies approximately in the terminator plane).

Perturbations due to C_{20}



Rotation of perigee distance ω for inclinations $i = 35^o, 63.4^o, 90^o$. For $i = 63.4^o$ the perigee stands still. For $i < 63.4^o$ the perigee moves ant-clockwise, for $i > 63.4^o$ clockwise.

The inclination $i = 63.4^o$ is called *critical inclination*.

Perturbations due to C_{20}

The standstill of the perigee for $i=63.4^o$ may be used to keep the perigee (or apogee) at a particular latitude.

Russia developed a system of telecommunication satellites (Molnja-type). The satellite orbits have

- Eccentricities of about $e=0.72$,
- SM axes of about *26550* km,
- A perigee distance of about $\omega=270^o$ and
- An inclination of $i=63.4^o$.

The satellites thus have their apogees at a Northern latitude of $i=63.4^o$ and a height of $h_{apo}=39'360$ km.

As the satellites move much slower near apogee than near perigee, the satellites spend about *8-10* above Northern latitudes (the revolution period is *12* h).

The Russian Tundra satellites (with ***e=0.27*** und and a revolution period of 1 day) also make use of the critical inclination. A permanent survey of the Northern hemisphere may be achieved with only three satellites.

Non-gravitational forces

As opposed to natural CB of the planetary system non-gravitational forces are much more important when dealing with artificial satellites.

The area/mass-ratio *A/m* of a satellite is the most important parameter to characterize the non-gravitational forces.

A is the cross section of the satellite for the perturbation considered (for drag normal to the velocity vector, for radiation pressure normal to the direction Sun->satellite), ***m*** is the satellite mass.

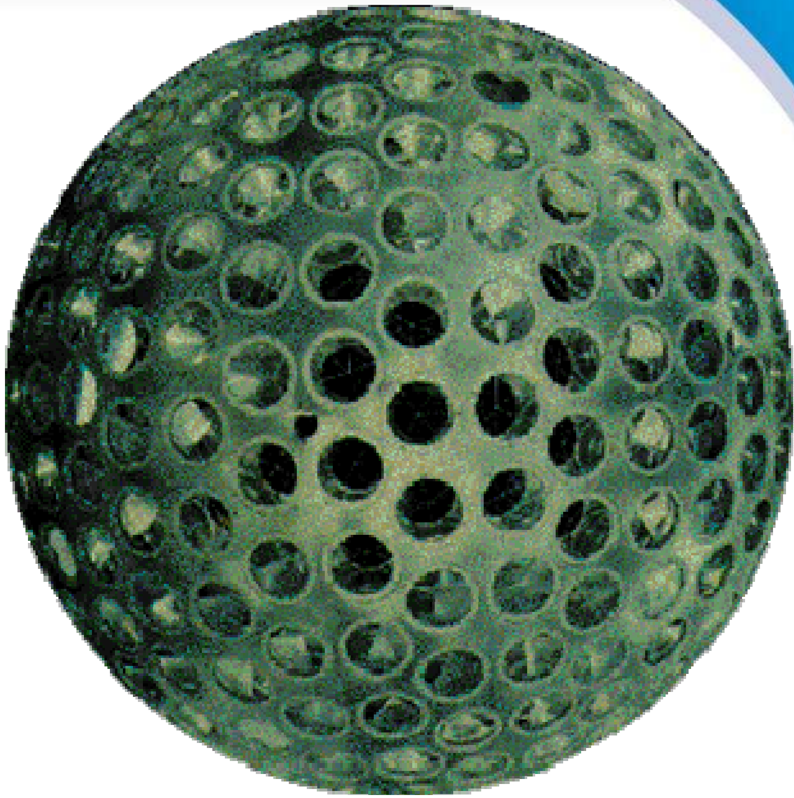
For the first generation of Earth science satellites the attempt was made to render *A/m* as small as possible..

A/m-ratios of a few important satellites:

Satellite	A/m [m^2/kg]
Lageos 1 and 2	0.0007
Starlette	0.001
GPS(Block II)	0.02
Moon	$1.3 \cdot 10^{-10}$

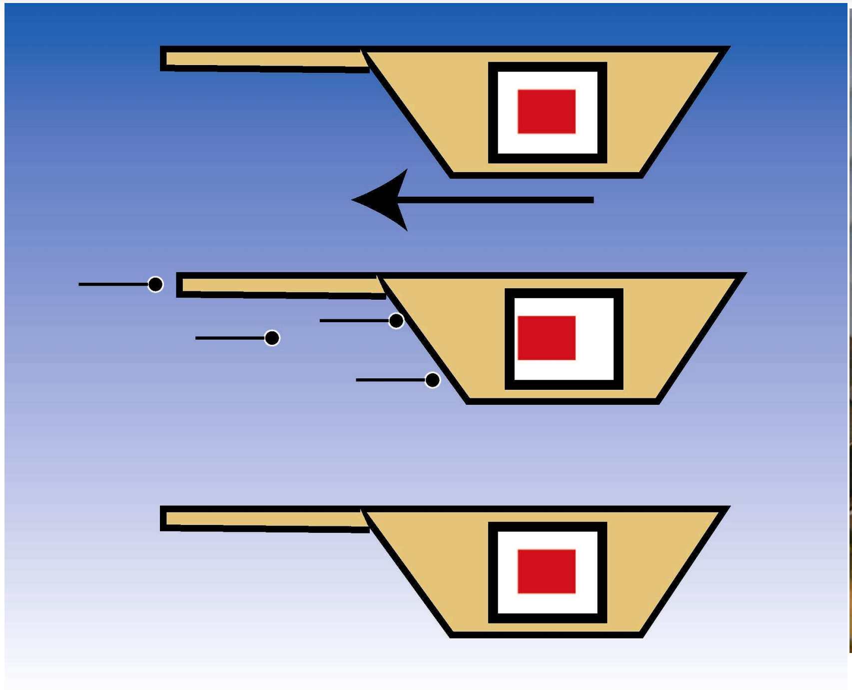
Non-gravitational forces

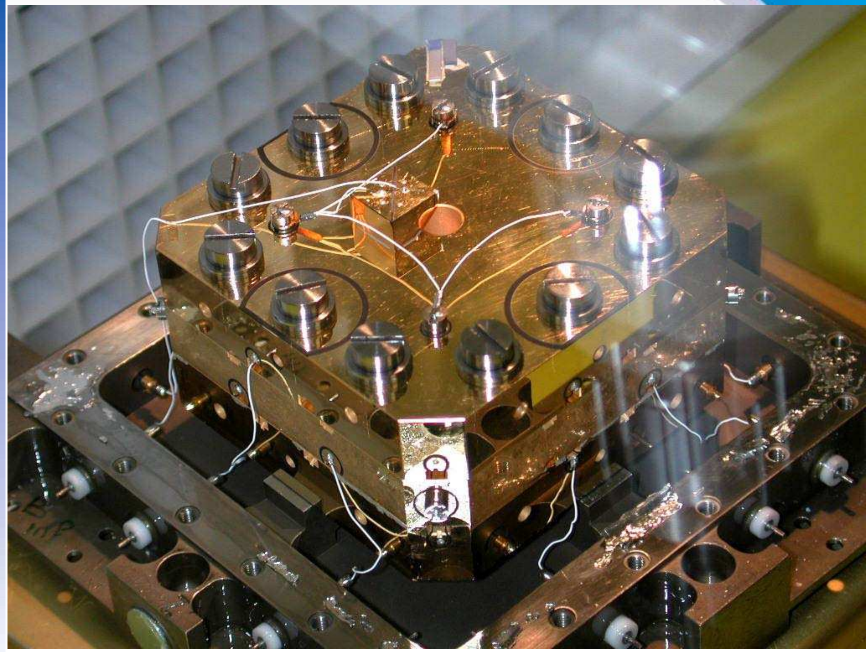
LAGEOS Parameters		
	LAGEOS-1	LAGEOS-2
Sponsor:	United States	United States & Italy
Expected Life:	many decades	many decades
Primary Applications:	geodesy	geodesy
COSPAR ID:	7603901	9207002
SIC Code:	1155	5986
NORAD SSC Code:	8820	22195
Launch Date:	May 4, 1976	October 22, 1992
RRA Diameter:	60 cm	60 cm
RRA Shape:	sphere	sphere
Reflectors:	426 corner cubes	426 corner cubes
Orbit:	circular	circular
Inclination:	109.84 degrees	52.64 degrees
Eccentricity:	0.0045	0.0135
Perigee:	5,860 km	5,620 km
Period:	225 minutes	223 minutes
Weight:	406.965 Kg	405.38 kg



Left: Characteristics of LAGEOS-1 und –2. Right: „Artist‘s view“ of Lageos-2. Our knowledge of the Earth’s gravity field stems in essence of these two satellites … where admittedly remarkable contributions go back to the times of Newton …).

Non-gravitational forces





Left: Principles of satellite accelerometry. The motion of a test mass in the shielded interior is measured (kept in place through electrostatic forces). Right: real satellite accelerometer.

Stability in time is critical → in any case one has to solve for one offset and one drift parameter per day (usually many more fudge parameters).

Non-gravitational forces: drag

Above a height of about *50* km the atmospheric density is sufficiently small to ignore turbulence.

Assuming that Earth's atmosphere rotates with the solid Earth, the perturbation due to drag may be calculated in a simple way – provided the atmospheric density $\rho(\boldsymbol{r})$ at the satellite position $\boldsymbol{r}$ is known.

Assuming furthermore that the satellite absorbs all molecules hitting it in $[t, t+\Delta t]$, the change of impuse Δp is calculated in the Earth-fixed system as:

$$\Delta \boldsymbol{p} = -\,\rho(\boldsymbol{r})\,A\,\dot{\boldsymbol{r}}'^2\,\Delta t\,\frac{\dot{\boldsymbol{r}}'}{|\dot{\boldsymbol{r}}'|}$$

From where we obtain the perturbing acceleration due to drag as:

$$\boldsymbol{a}_d = -\,\rho(\boldsymbol{r})\,\frac{A}{m}\,\dot{\boldsymbol{r}}'^2\,\frac{\dot{\boldsymbol{r}}'}{|\dot{\boldsymbol{r}}'|}$$

Non-gravitational forces: drag

Real satellite is more complicated because not all molecules are absorbed by the satellite, but reflected from its surface, which implies:

- The resulting acceleration must not be precisely anti-parallel to the velocity vector.
- For the resulting acceleration we introduce an empirical pre-factor *C,* accounting approximately for the difference between absorbed and reflected molecules.

C may be calculated for special satellites (for spherically symmetric satellites we have *C=2*, i.e., drag is independent of the fraction absorbed/reflected molecules.

The general formula for drag reads as:

$$\boldsymbol{a}_d = -\frac{C}{2}\,\rho(\boldsymbol{r})\,\frac{A}{m}\,\dot{\boldsymbol{r}}'^2\,\frac{\dot{\boldsymbol{r}}'}{|\dot{\boldsymbol{r}}'|}$$

where: $2 \leq C \leq 2.5$

Non-gravitational forces: drag

Locally, the density $\rho(h)$ of the atmosphere may be approximated by the *barometric height formula*:

$$\rho(h) = \rho_0 \; e^{-\frac{h-h_0}{H_0}}$$

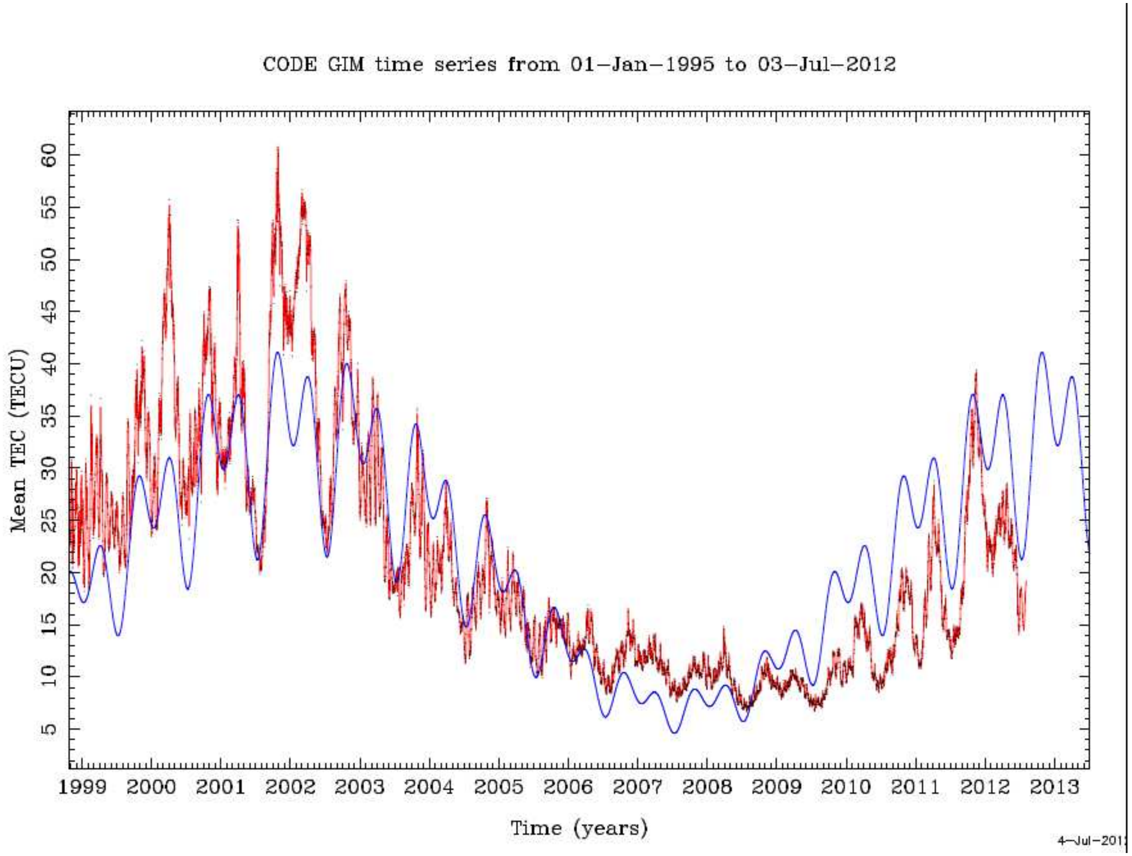
h_0 is reference height, H_0 the scale height.

Even for low eccentricity orbits the difference $h_{apo} - h_{per} >> H_0$, implying that usually one cannot work with one barometric formula over one revolution of the satellite motion.

Moreover $\rho(h)$ is latitude- and longitude-dependent and has seasonal and Sun-cycle variations.

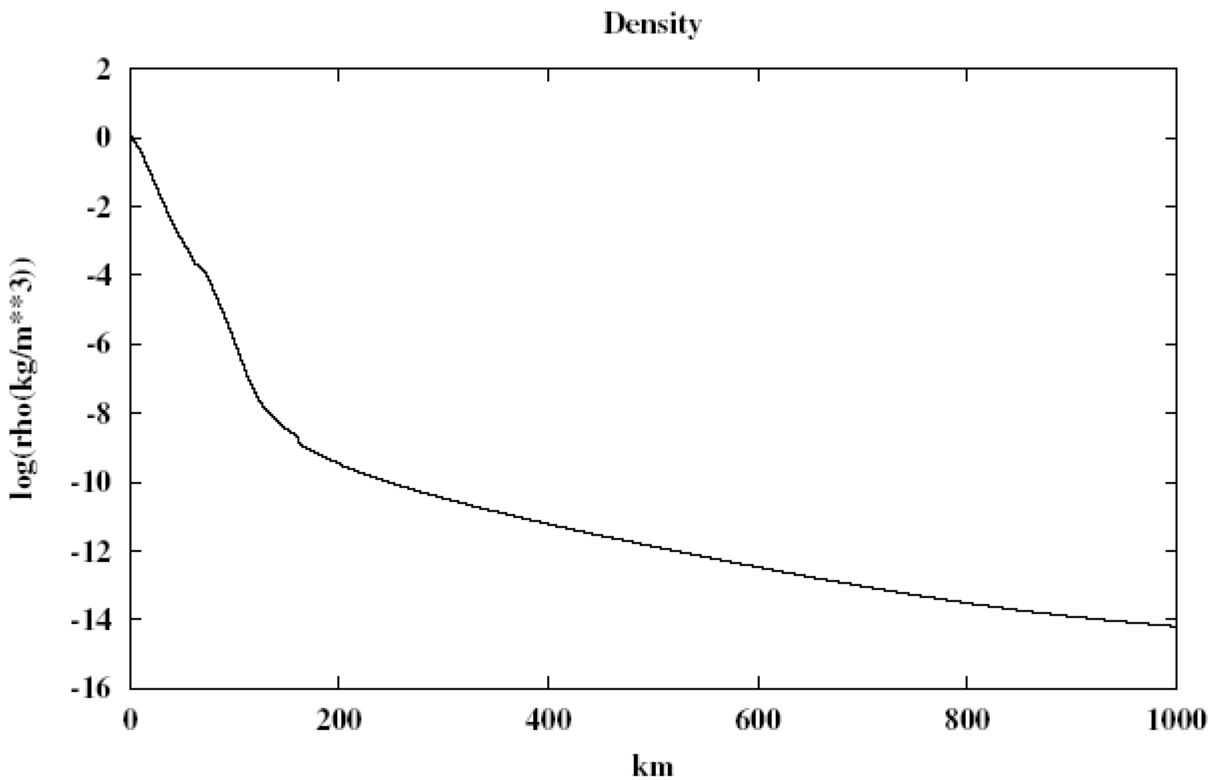
Even detailed models do not allow it to introduce drag as a „known“ force. For high-accuracy applications one has to solve at least for a scaling factor.

Non-gravitational forces: drag



Mean electron content in the upper atmosphere as estimated by CODE (Center for Orbit Determination in Europe). Note the length of the most recent solar cycle.

Non-gravitational forces: drag



Mean atmospheric density from the MSIS-model.

Drag may be neglected above a height of about *2000* km.

Non-gravitational forces: drag

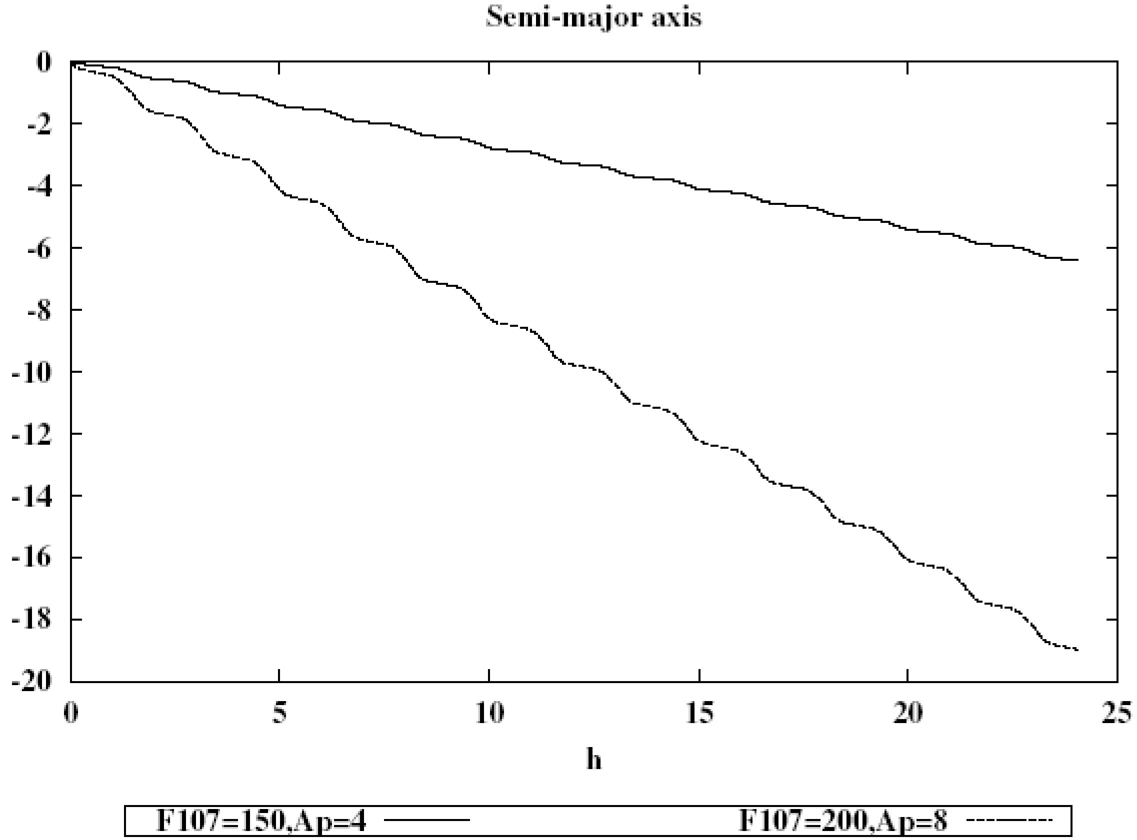
Let us use the orbit characteristics of GPS/MET, a test satellite of the mid 1990s to study atmosphere sounding with GPS, to study the impact of drag.

The orbit characteristics and the A/m ratio for GPS/MET are:

Table 3.5. Characteristics of GPS/MET in 1995

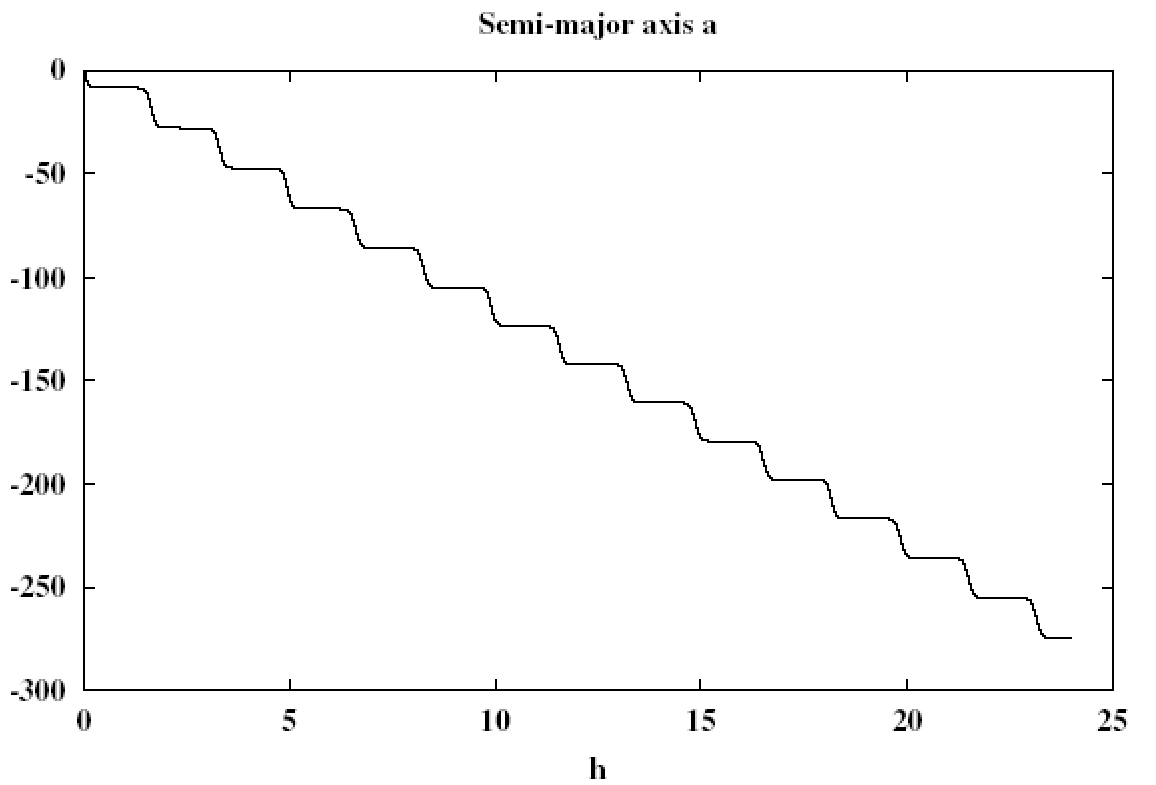
Satellite/Orbit Property	Numerical Value
a	7100 km
e	0.0
i	70°
A/m	0.02

Non-gravitational forces: drag



Decay of the semi-major -axis *a* of GPS/MET making different (but realistic) assumptions concerning the atmospheric density. Bottom line: The sm-axis decreases by several m/day.

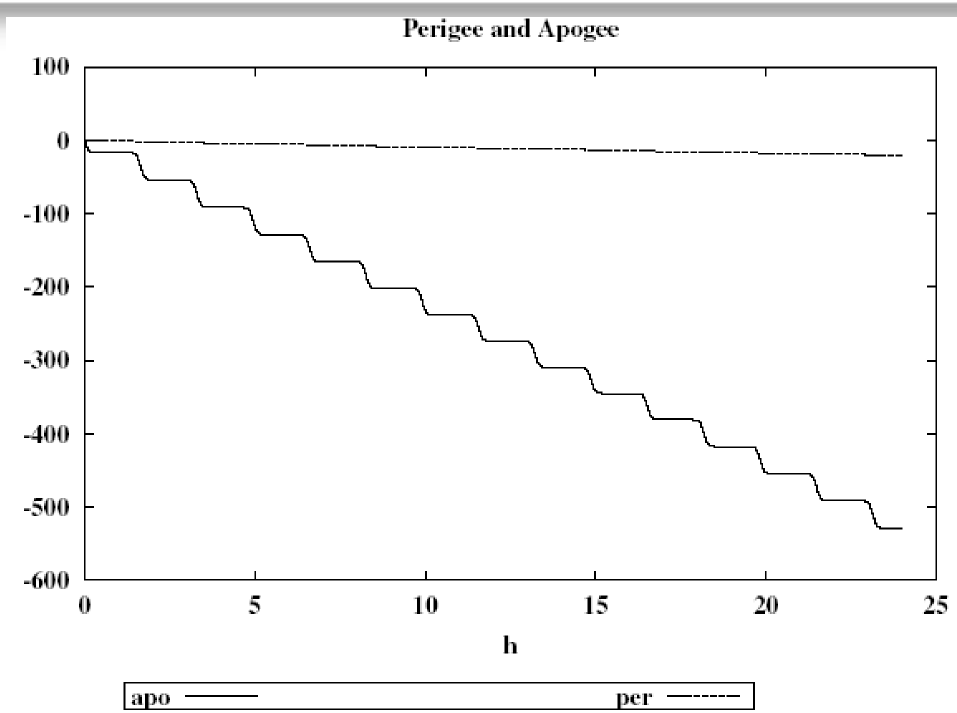
Non-gravitational forces: drag



With an eccentricity of *e=0.05* instead of *e=0.02*, the satellite would have penetrated more deeply into the atmosphere and the perturbation would have been much stronger!

For satellites with “higher” eccentricities, drag outside the perigee is almost irrelevant.

Non-gravitational forces: drag



Decrease of Perigee and Apogee of GPS/MET (with *e=0.05*) over one day:

- There perigee barely changes, whereas the apogee decreases by several hundred m per day! →
- Orbits are getting more circular due to drag.
- Drag cleans the atmosphere < 1000 km from LEOs and space debris!

Non-gravitational forces: rpr

Quantum theory says that a photon with frequency ν and wave-length $\lambda = c/\nu$ carries an energy $E = h\nu$ and a momentum of $p = h\nu/c\ \boldsymbol{e}$, where $h = 6.62\ 10^{-34}$ Js is Planck's constant, $\boldsymbol{e}$ is the unit vector of propagation of the photon.

In practice it is a difficult task to calculate the transfer of momentum on satellite with complicated shape in a general radiation field. The principles underlying the task are, however, simple:

- The satellite surface is subdivided into small surface elements.
- For each surface element and for each frequency one calculates the number of incident photons and the associated momentum.
- If all photons are absorbed, the resulting momentum equals the sum of all incident photon momentums.
- For the fraction of the photons reflected, the resulting momentum is the difference of the photon momentums of all reflected and all incident photons.

Non-gravitational forces: rpr

Under the following assumptions *rpr* acting on a satellite may be calculated easily:

- The Sun is the only radiation source
- The radiation is assumed to be parallel at the satellite
- The entire radiation is absorbed by the satellite:

$$\boldsymbol{a}_{\mathrm{rad}} = \frac{\tilde{C}}{2} \frac{A_{\dagger}^{2}}{|\boldsymbol{r} - \boldsymbol{r}_{\odot}|^{2}} \frac{S}{c} \frac{A}{m} \frac{\boldsymbol{r} - \boldsymbol{r}_{\odot}}{|\boldsymbol{r} - \boldsymbol{r}_{\odot}|}$$

Where

S is the solar constant (at 1 AU) (=incident Energy on an area of 1 m^2 per s normal to the direction Sun →area).

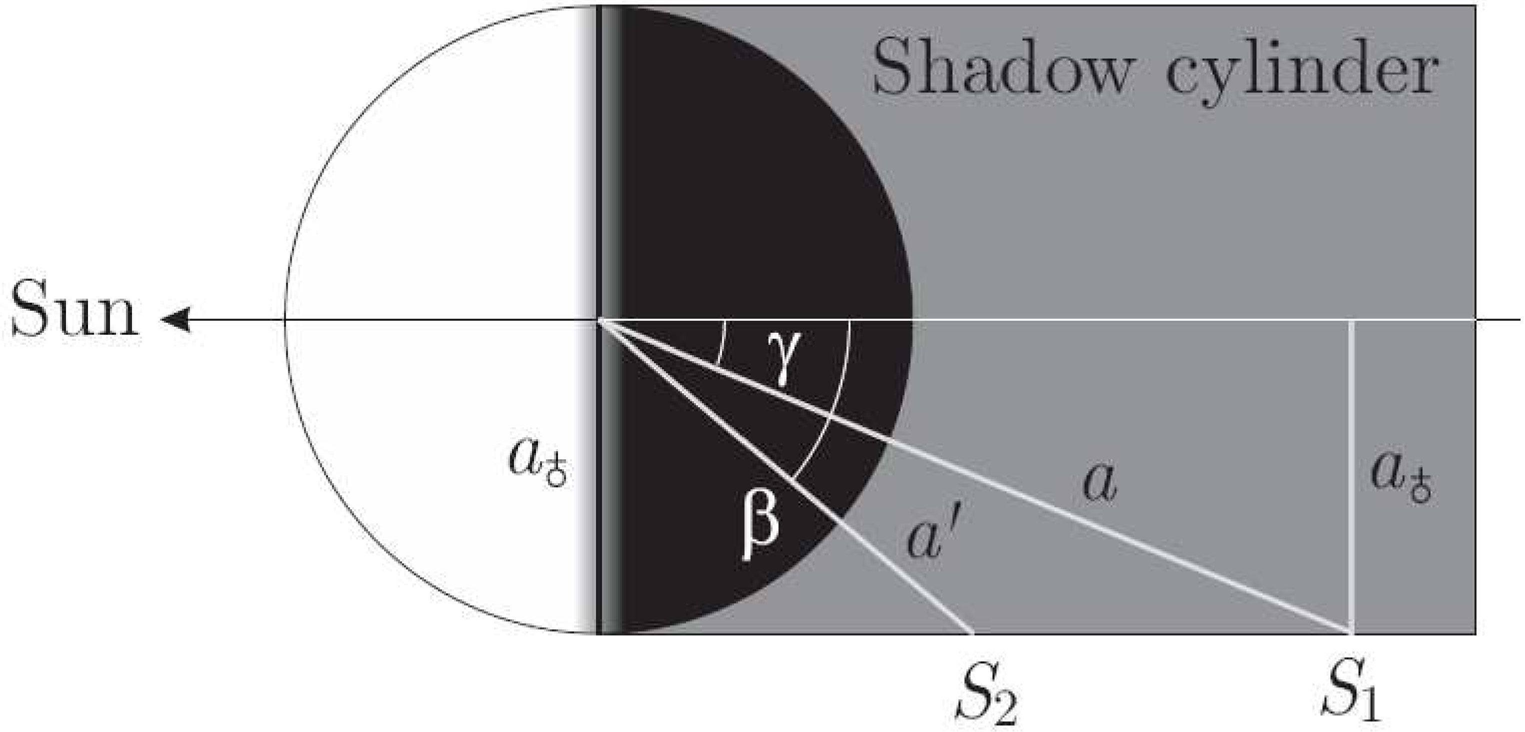
A.. Is the Astronomical Unit,

r is the geocentric position vector of the satellite, and

$\boldsymbol{r}_{\mathbf{o}}$ is the geocentric position vector of the Sun.

The factor would be *C=2*, if all photons were absorbed. Usually one has $2 < C < 2.5$.

Non-gravitational forces: rpr



Modeling *rpr* is encumbered by the fact that rpr has to be switched of, when the satellite enters the Earth's shadow.

The transition light <-> shadow is relatively rapid. This is a problem for analytical and numerical procedures.

Non-gravitational forces: rpr

On the average over one revolution *rpr* barely changes the elements *i* and Ω (except in shadow periods).

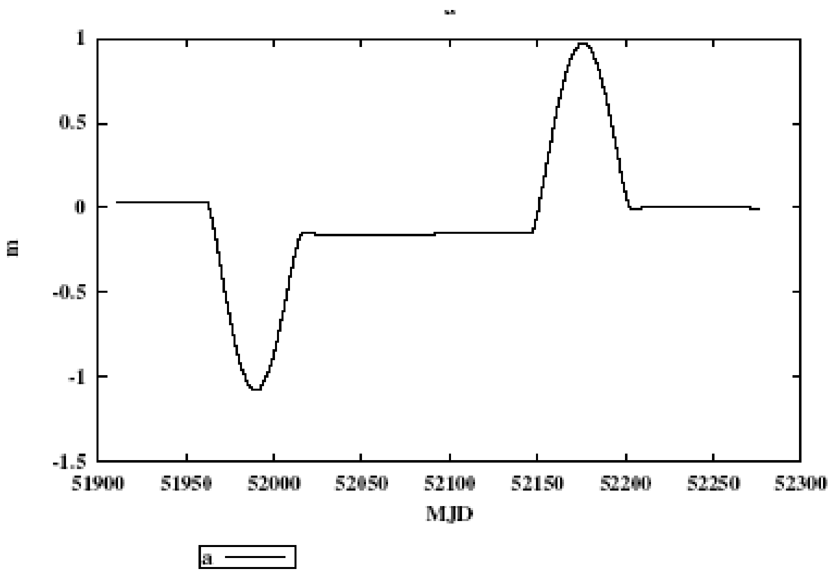
Eccentricity *e* and perigee distance ω are effected heavily. Only when averaging over the draconitic year of a particular satellite, the impact is averaged out.

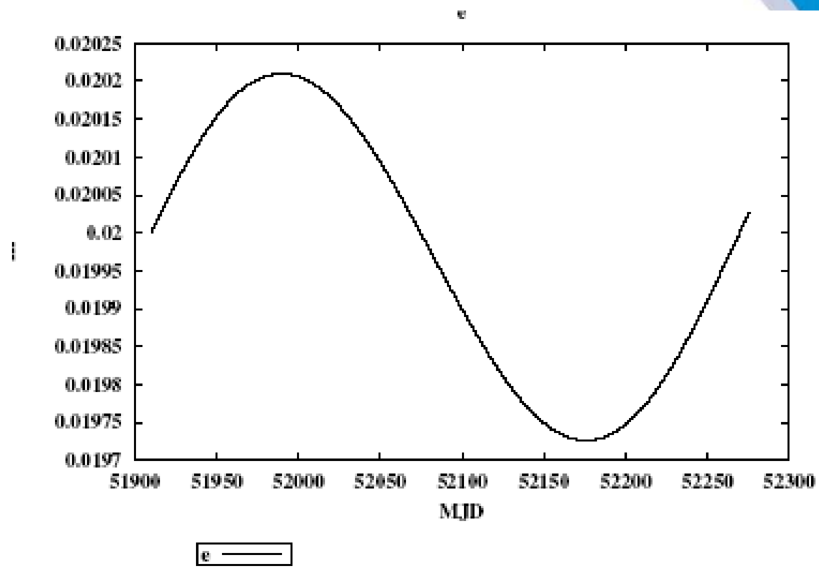
Depending on the *A/m*-ratio very large long-period effects may result. Such effects occurred for *Space Debris.* Periodic changes up to $\Delta e = 0.7$ occur.

The duration of shadow periods heavily depends on the orbit characteristics. Up to half of the orbit may be in shadow for LEO satellites, for high-orbiting satellites (e.g., geostationary or GNSS-type satellites the shadow period is of the order of a few percent of the revolution time).

For GNSS satellites, orbiting the Earth at a height of about *20‘000* km, the maximum shadow duration is about one hour.

Non-gravitational forces: rpr





Impact of *rpr* on the sm-axis *a* and on the eccentricity *e* of a GPS-satellite (over the period of one year).

Averaged over one orbital period the *rpr* perturbations in *a* are zero (except during the shadow periods).

The eccentricity *e* changes smoothly over the year.

Non-gravitational forces: rpr

Apart from solar (or direct) *rpr* the following effects have to be taken into account for accurate orbits:

- *rpr* due to solar radiation reflected/re-emitted by the Earth (Albedo-rpr) as well as
- *rpr* due to solar radiation reflected/re-emitted by the Moon.

Within the IGS one has even been able to identify (missing) *rpr* effects during lunar eclipses!

In summary one may say that *rpr* must be carefully modeled for LEOs, MEOs, GEOs.

Precise *rpr* modeling currently is the most important accuracy-limiting effect for GNSS orbits. It would be interesting to see a future generation of GNSS with accelerometers.

Forces acting satellites in overview

Perturbation	Acceleration	Orbit Error after one Day		
		Radial	Along Track	Out of Plane
	[m/s^2]	[m]	[m]	[m]
$\frac{1}{r^2}$-Term	8.42	"∞"	"∞"	"∞"
Oblateness	$1.5 \cdot 10^{-2}$	60000	400000	900000
Atmospheric Drag	$7.9 \cdot 10^{-7}$	150	8900	1.5
Higher Terms of the Earth's Grav. Field	$2.5 \cdot 10^{-4}$	550	3400	820
Lunar Attraction	$5.4 \cdot 10^{-6}$	2	45	2
Solar Attraction	$5.0 \cdot 10^{-7}$	1	38	15
Direct Rad. Pressure	$9.7 \cdot 10^{-8}$	10	24	0
Solid Earth Tides	$1.1 \cdot 10^{-7}$	0.2	13	1
y-bias	$1.0 \cdot 10^{-9}$	0.1	4.7	0.0

LEO Orbits: Orbit errors after one day when *not* taking into account particular perturbations.

Forces acting satellites in overview

Perturbation	Acceleration	Orbit Error after one Day		
		Radial	Along Track	Out of Plane
	[m/s^2]	[m]	[m]	[m]
$\frac{1}{r^2}$-Term	0.57	"∞"	"∞"	"∞"
Oblateness	$5.1 \cdot 10^{-5}$	2750	32000	15000
Lunar Attraction	$4.5 \cdot 10^{-6}$	400	1800	30
Solar Attraction	$2 \cdot 10^{-6}$	200	1200	400
Higher Terms of the Earth's Grav. Field	$4.2 \cdot 10^{-7}$	60	440	10
Direct Rad. Pressure	$9.7 \cdot 10^{-8}$	75	180	5
y-bias	$1.0 \cdot 10^{-9}$	0.9	8.1	0.3
Solid Earth Tides	$5.0 \cdot 10^{-9}$	0.0	0.4	0.0
Atmospheric Drag	—	—	—	—

GNSS Orbits: Orbit errors after one day when *not* taking into account particular perturbations.