



4ème Ecole d'été du GRGS

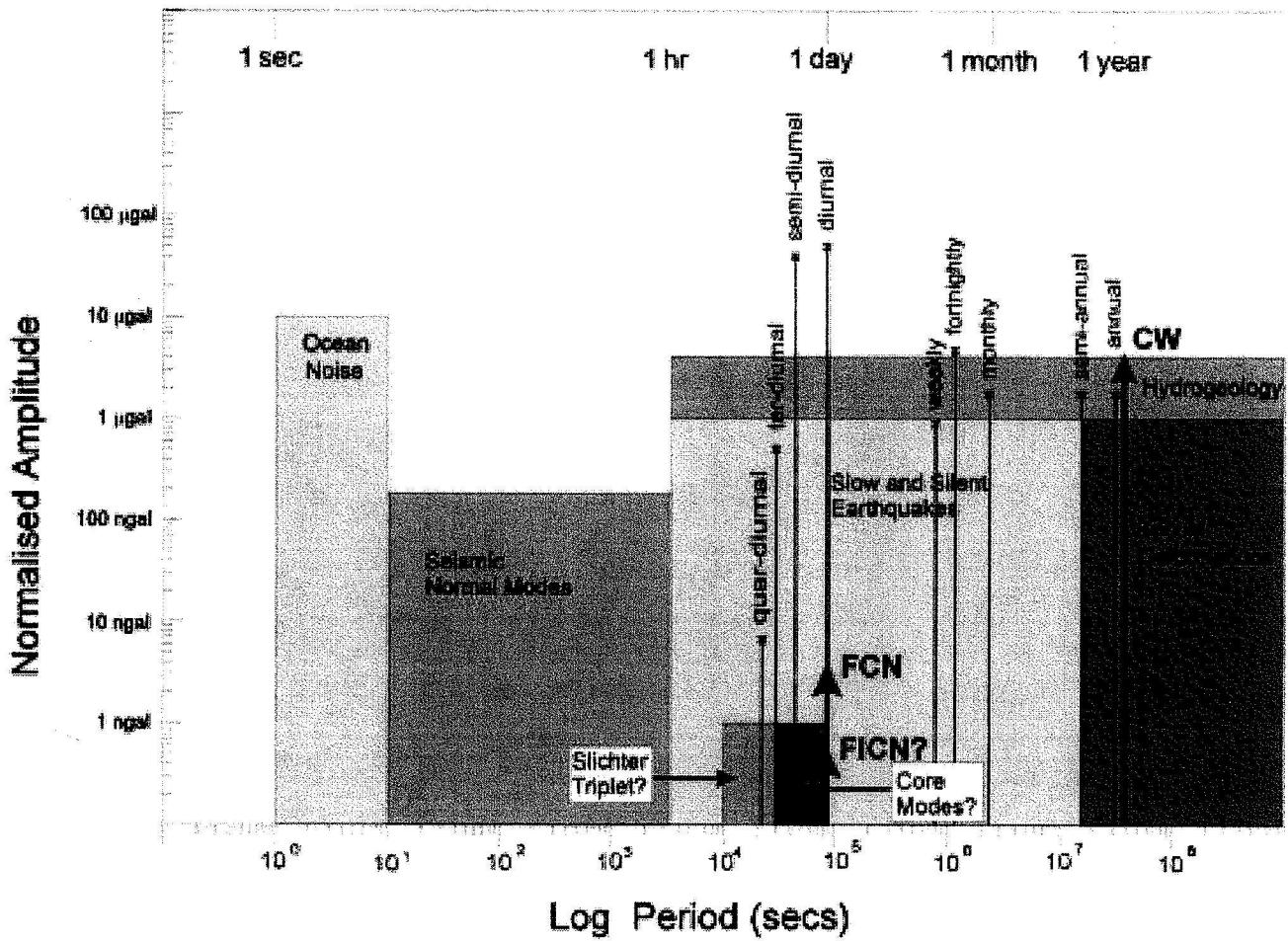
Déformations élastiques et viscoélastiques de la Terre

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Déformations élastiques et viscoélastiques de la Terre



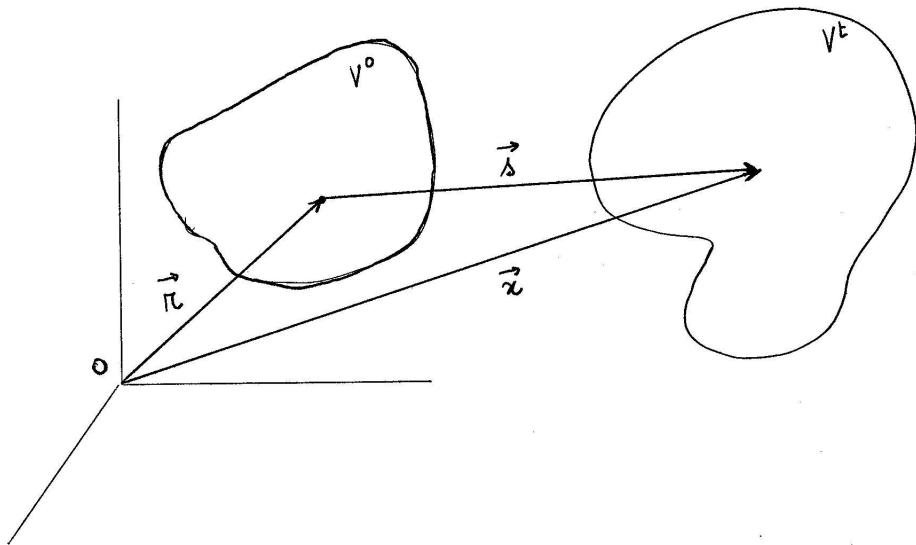
(Hinderer et Crossley 2000)

Déformations élastiques et viscoélastiques de la Terre

1. Rappels de mécanique des milieux continus
2. Déformation d'une Terre élastique
 - (a) Oscillations libres
 - (b) Déformation forcée
3. Déformation d'une Terre viscoélastique

- 1. Rappels de mécanique des milieux continus**
2. Déformation d'une Terre élastique
 - (a) Oscillations libres
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Loi du mouvement



r = position initiale de la particule matérielle r

x = position de r à l'instant t

s = déplacement de r

$x(r, t) = r + s(r, t)$: loi du mouvement

$x(r, t)$ invertible $\Rightarrow r = r(x, t)$

En $t = 0$: $x(r, 0) = r$

Variables lagrangiennes et variables eulériennes

Soit une quantité physique:

$F(\mathbf{x}, t)$: description **eulérienne (spatiale)**

$f(\mathbf{r}, t)$: description **lagrangienne (matérielle)**

Passage de l'une à l'autre:

$$F(\mathbf{x}, t) = f[\mathbf{r}(\mathbf{x}, t), t]$$

$$f(\mathbf{r}, t) = F[\mathbf{x}(\mathbf{r}, t), t]$$

Variations lagrangiennes et eulériennes

$$F(\mathbf{x}, t) = F(\mathbf{x}, 0) + \Delta F(\mathbf{x}, t)$$

$$\Delta F(\mathbf{x}, t) = \Delta f[\mathbf{r}(\mathbf{x}, t), t]$$

ΔF ou Δf : variation eulérienne

$$f(\mathbf{r}, t) = f(\mathbf{r}, 0) + \delta f(\mathbf{r}, t)$$

$$\delta f(\mathbf{r}, t) = \delta F[\mathbf{x}(\mathbf{r}, t), t]$$

δF ou δf : variation lagrangienne

Perturbations infinitésimales

Hypothèses:

- $|s|$ petit
- $\Delta F(\mathbf{x}, t) \ll F(\mathbf{x}, 0)$
- $\delta f(\mathbf{r}, t) \ll f(\mathbf{r}, 0)$

⇒ Linéarisation

$$\Rightarrow \delta f(\mathbf{r}, t) \simeq \Delta f(\mathbf{r}, t) + s(\mathbf{r}, t) \cdot \nabla_{\mathbf{r}} f(\mathbf{r}, 0)$$

Vecteur dérivant d'un potentiel

$$\mathbf{V}(\mathbf{x}, t) = -\nabla_{\mathbf{x}} \Phi(\mathbf{x}, t)$$

$$\mathbf{V}(\mathbf{x}, 0) + \Delta \mathbf{V}(\mathbf{x}, t) = -\nabla_{\mathbf{x}} [\Phi(\mathbf{x}, 0) + \Delta \Phi(\mathbf{x}, t)]$$

$$\Rightarrow \Delta \mathbf{V}(\mathbf{x}, t) = -\nabla_{\mathbf{x}} \Delta \Phi(\mathbf{x}, t)$$

mais

$$\delta \mathbf{v}(\mathbf{r}, t) \simeq -\nabla_{\mathbf{r}} \delta \phi(\mathbf{r}, t) + \nabla_{\mathbf{r}} s(\mathbf{r}, t) \cdot \nabla_{\mathbf{r}} \phi(\mathbf{r}, 0)$$

où

$$(\nabla_{\mathbf{r}} s)_{ij} = \frac{\partial s_j}{\partial r_i}$$

Variations de volume, d'aire et de longueur

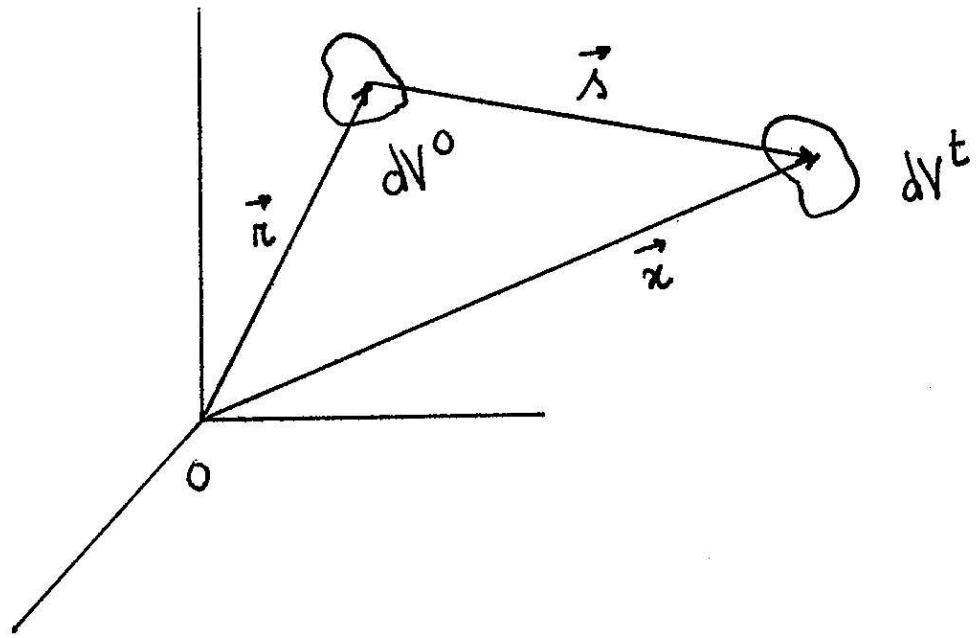
Gradient de la déformation:

$$\begin{aligned}f_{ij} &= \frac{\partial x_i}{\partial r_j} \\&= \delta_{ij} + \frac{\partial s_i}{\partial r_j}\end{aligned}$$

ou

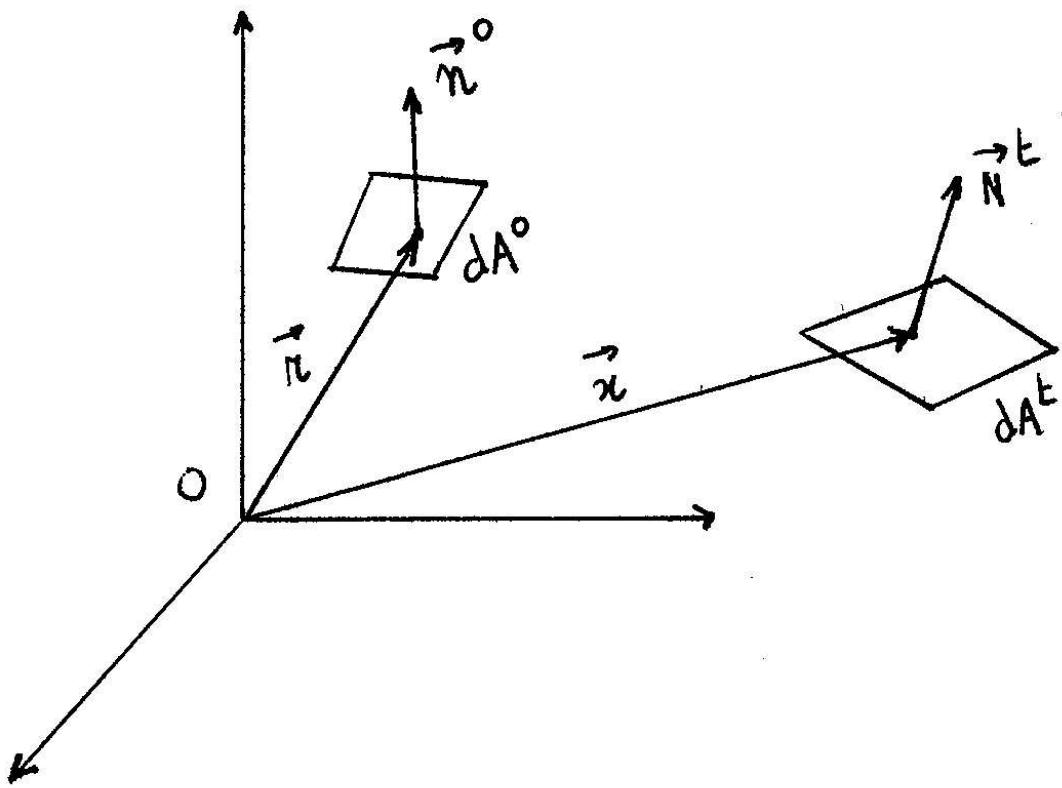
$$\begin{aligned}\mathbf{f}^T &= \nabla_{\mathbf{r}} \mathbf{x} \\&= \mathbf{I} + \nabla_{\mathbf{r}} \mathbf{s}\end{aligned}$$

Variation de volume



$$\begin{aligned} J &= \frac{dV^t}{dV^0} \\ &= \text{dtm}(f) \\ &\simeq 1 + \nabla \cdot s \end{aligned}$$

Variation d'aire

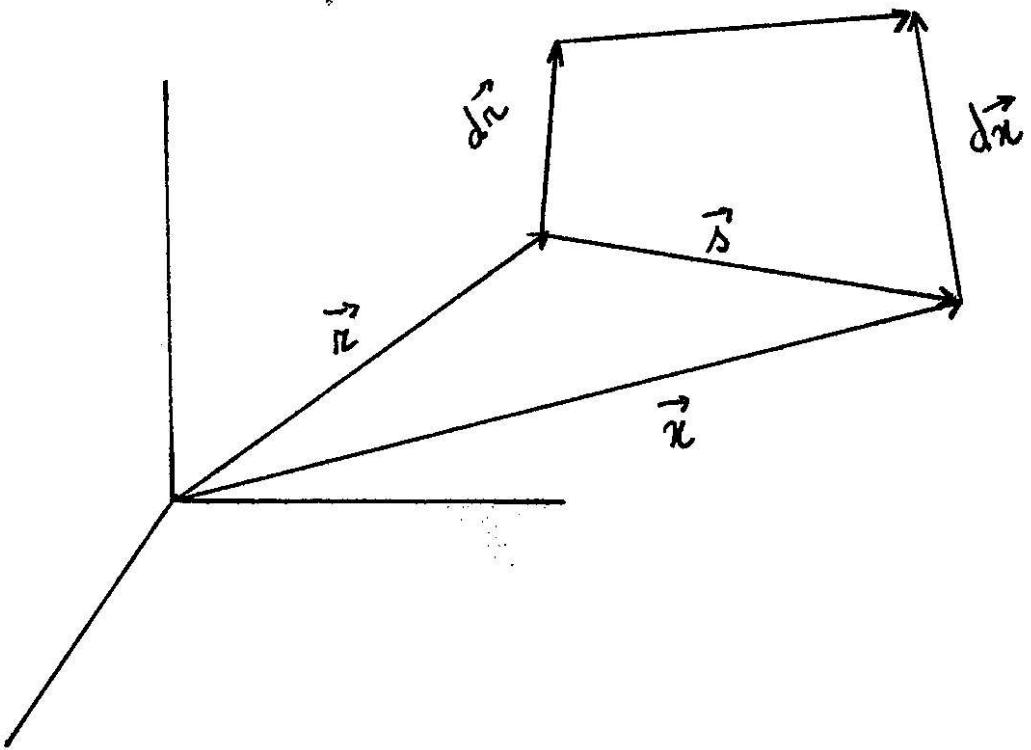


$$dA^t \mathbf{N} = J dA^0 \mathbf{n}^0 \cdot f^{-1}$$

$$\Rightarrow \mathbf{n}(\mathbf{r}, t) \simeq \mathbf{n}^0 \left(1 + \mathbf{n}^0 \cdot \nabla \mathbf{s} \cdot \mathbf{n}^0 \right) - \nabla \mathbf{s} \cdot \mathbf{n}^0$$

$$dA^t \simeq dA^0 \left(1 + \nabla \cdot \mathbf{s} - \mathbf{n}^0 \cdot \nabla \mathbf{s} \cdot \mathbf{n}^0 \right)$$

Variation de longueur



$$dx^2 - dr^2 = 2dr \cdot e \cdot dr$$

e = tenseur des déformations

$$\begin{aligned} e &= \frac{1}{2} (f^\top \cdot f - I) \\ &= \frac{1}{2} [\nabla s + (\nabla s)^\top + \nabla s \cdot (\nabla s)^\top] \\ &\simeq \frac{1}{2} [\nabla s + (\nabla s)^\top] \end{aligned}$$

Contraintes

Tenseur des contraintes de Cauchy \mathbf{T} :

$$d\mathbf{F}(\mathbf{x}, t) = \mathbf{N} \cdot \mathbf{T} dA^t$$

Premier tenseur de Piola-Kirchoff $\boldsymbol{\tau}$:

$$d\mathbf{F}[\mathbf{x}(\mathbf{r}, t), t] = \mathbf{n}^0 \cdot \boldsymbol{\tau} dA^0$$

$$\boldsymbol{\tau} = J(f)^{-1} \cdot \mathbf{t}$$

Equations fondamentales

Inconnues:

Vitesse: $\mathbf{V}(\mathbf{x}, t) = \frac{\partial \mathbf{x}}{\partial t}|_{\mathbf{r}}$

Densité: $\rho^E(\mathbf{x}, t)$

Contraintes: $\mathbf{T}(\mathbf{x}, t)$

Gravité propre: $\mathbf{G}(\mathbf{x}, t) = -\nabla_{\mathbf{x}}\Phi(\mathbf{x}, t)$

(si effets thermiques: Energie interne U , Courant de chaleur \mathbf{Q} , Entropie S , Température Θ)

Conservation de la masse

$$\frac{D\rho^E}{Dt} + \rho^E \nabla_x \cdot V = \frac{\partial \rho^E}{\partial t} + \nabla_x \cdot (\rho^E V) = 0$$

Conservation de la quantité de mouvement

$$\rho^E \frac{D\mathbf{V}}{Dt} = \nabla_x \cdot \mathbf{T} + \rho^E \mathbf{G} + \mathbf{F}_{\text{ext}}$$

$$\rho^0 \frac{\partial^2 \mathbf{s}}{\partial t^2} = \nabla_{\mathbf{r}} \cdot \boldsymbol{\tau} + \rho^0 \mathbf{g} + \mathbf{f}_{\text{ext}}$$

Conservation du moment cinétique

$$\mathbf{T} = \mathbf{T}^\top$$

Equation de Poisson

$$\nabla^2 \Phi = 4\pi G \rho^E$$

Equation constitutive

$$\mathbf{T} = \mathcal{F} \left(e(t')_{-\infty}^t, e, \frac{\partial e}{\partial t}, \dots \right)$$

Si effets thermiques:

Conservation de l'énergie (1er principe)

$$\rho^E \frac{DU}{Dt} = T : D + R - \nabla_x \cdot Q$$

$$D = \frac{1}{2} [\nabla V + (\nabla V)^T]$$

R = source de chaleur

Bilan d'entropie (2ème principe)

$$\rho^E \frac{DS}{Dt} \geq -\nabla_x \cdot \frac{Q}{\Theta} + \frac{R}{\Theta}$$

Equation d'état

$$U = U(\rho^E, S \dots)$$

$$\Theta = \left. \frac{\partial U}{\partial S} \right|_{\rho^E, \dots}$$

Equation constitutive

$$Q = Q(D, \Theta, \nabla \Theta, \dots)$$

Conditions de continuité

Interface solide-fluide non visqueux:

$$[V \cdot N]_-^+ = 0$$

Interface solide-solide ou solide-fluide visqueux:

$$[V]_-^+ = 0$$

Partout:

$$[T \cdot N]_-^+ = 0$$

$$[\Phi]_-^+ = 0$$

$$[\nabla \Phi]_-^+ = 0$$

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Équilibre hydrostatique

$$\mathbf{V} = 0$$

$$\frac{\partial \cdot}{\partial t} = 0$$

$$\mathbf{T} = -p^0 \mathbf{I}$$

$$\Rightarrow \nabla p^0 = -\rho^0 \nabla \phi^0$$

$$\nabla^2 \phi^0 = 4\pi G \rho^0$$

⇒ Figure sphérique: $\rho^0(r)$

Si $r < R$:

$$\phi^0(r) = -\frac{4\pi G}{r} \int_0^r \rho^0(r') r'^2 dr' - 4\pi G \int_r^R \rho^0(r') r'^2 dr'$$

$$\frac{dp^0}{dr} = -\rho^0 \frac{d\phi^0}{dr}$$

Si $r > R$:

$$\phi^0(r) = -\frac{4\pi G}{r} \int_0^R \rho^0(r') r'^2 dr'$$

Perturbation infinitésimale de l'équilibre hydrostatique (en variables lagragiennes)

Equation de continuité

$$\Delta \rho^L = -\nabla \cdot (\rho^0 \mathbf{s})$$

Equation de Poisson

$$\nabla^2 \Delta \phi = 4\pi G \Delta \rho^L$$

Equation constitutive

Elasticité linéaire et isotrope

$$\begin{aligned} t &= -p^0 \mathbf{I} + \lambda^0 (\nabla \cdot \mathbf{s}) \mathbf{I} + 2\mu^0 \mathbf{e} \\ &= -p^0 \mathbf{I} + \delta t \end{aligned}$$

λ^0 et μ^0 : paramètres de Lamé

Conservation de la quantité de mouvement

$$\rho^0 \frac{\partial^2 \mathbf{s}}{\partial t^2} = \nabla \cdot \delta \mathbf{t} - \rho^0 \nabla(\Delta\phi) + \rho^0 (\nabla \cdot \mathbf{s}) \nabla\phi^0 - \rho^0 \nabla (\mathbf{s} \cdot \nabla\phi^0) + \mathbf{f}_{\text{ext}}$$

Hypothèse:

$$\mathbf{f}_{\text{ext}} = -\rho^0 \nabla\phi_{\text{ext}}$$

On pose:

$$\phi_1 = \Delta\phi + \phi_{\text{ext}}$$

Modèle de Terre sphérique PREM

(Dziewonski et Anderson 1981)

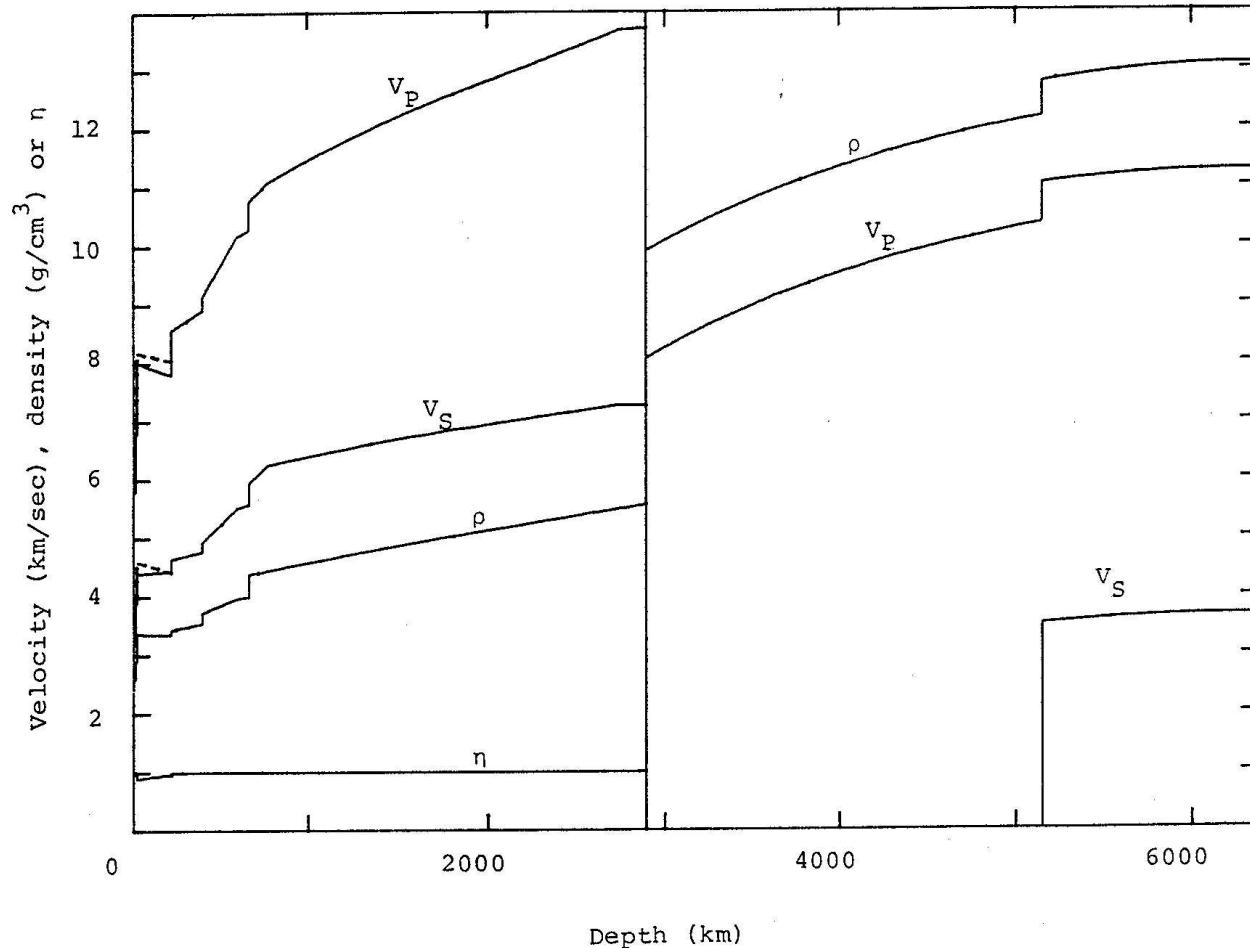


Fig. 8. The PREM model. Dashed lines are the horizontal components of velocity. Where η is 1 the model is isotropic. The core is isotropic.

REM

<http://mahi.ucsd.edu/Gabi/rem.html>

Séparation des variables

en coordonnées sphériques

1. Mouvements **oscillatoires** (en $e^{i\omega t}$):

$$\Rightarrow \frac{\partial}{\partial t} = i\omega$$

2. Décomposition de Helmholtz

$$\begin{aligned}\mathbf{s}(\mathbf{r}, \omega) &= U\mathbf{e}_r + \nabla \times (W\mathbf{r}) + \mathbf{e}_r \times [\nabla \times (V\mathbf{r})] \\ &= U\mathbf{e}_r + \left(\frac{\partial V}{\partial \theta} + \frac{1}{\sin \theta} \frac{\partial W}{\partial \varphi} \right) \mathbf{e}_\theta \\ &\quad + \left(\frac{1}{\sin \theta} \frac{\partial V}{\partial \varphi} - \frac{\partial W}{\partial \theta} \right) \mathbf{e}_\varphi \\ &= U\mathbf{e}_r + \nabla_S V - \mathbf{e}_r \times \nabla_S W\end{aligned}$$

où

$$\nabla_S = \mathbf{e}_\theta \frac{\partial}{\partial \theta} + \mathbf{e}_\varphi \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi}$$

Déplacement **sphéroïdal**: $U\mathbf{e}_r + \nabla_S V$

Déplacement **toroïdal**: $-\mathbf{e}_r \times \nabla_S W$

Traction:

$$\mathbf{e}_r \cdot \delta \mathbf{t} = R\mathbf{e}_r + \nabla_S S - \mathbf{e}_r \times \nabla_S T$$

Variation de gravité:

$$\psi = \frac{\partial \phi_1}{\partial r} + 4\pi G \rho^0 U$$

\Rightarrow 8 scalaires: $U, V, W, R, S, T, \phi_1, \psi$

3. Décomposition en harmoniques sphériques:

$$Y_\ell^m = (-1)^m \sqrt{\frac{(2\ell+1)}{4\pi} \frac{(\ell-|m|)!}{(\ell+|m|)!}} P_\ell^m(\cos \theta) e^{im\varphi}$$

$$\ell = 0, 1, 2, \dots \infty$$

$$m = -\ell, -\ell+1, \dots, \ell-1, \ell$$

$m \geq 0$:

$$P_\ell^m(x) = \frac{1}{2^\ell \ell!} \sqrt{(1-x^2)^m} \frac{d^{\ell+m}}{dx^{\ell+m}} (x^2 - 1)^\ell$$

$m < 0$:

$$P_\ell^{-m}(x) = \frac{(\ell+m)!}{\ell-m)!} P_\ell^m(x)$$

Orthonormalité des harmoniques sphériques:

$$\int_0^{2\pi} \int_0^\pi Y_\ell^m(\theta, \varphi) Y_{\ell'}^{m'*}(\theta, \varphi) \sin \theta d\theta d\varphi = \delta_{\ell\ell'} \delta_{mm'}$$

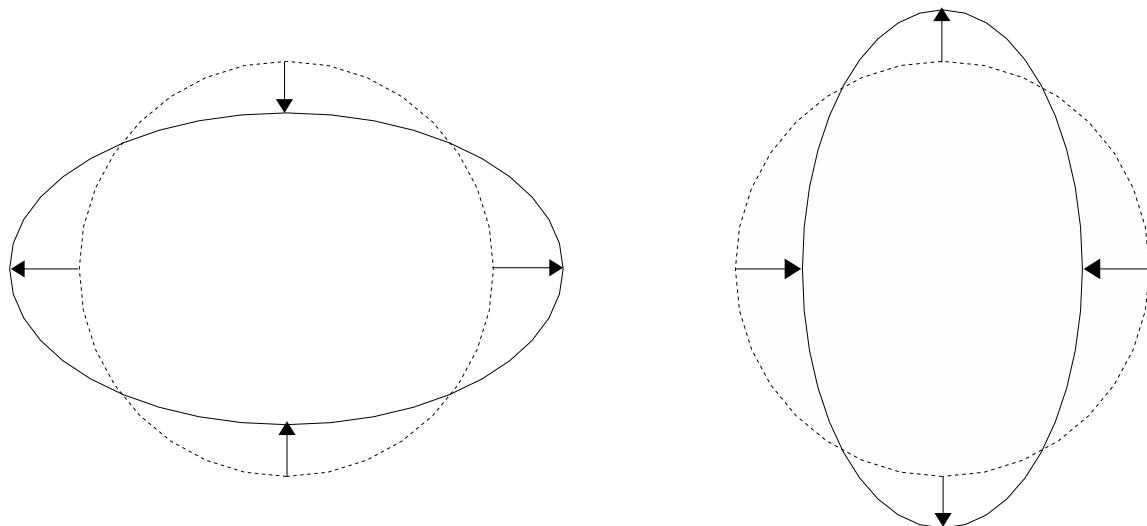
Pour $U, V, W, R, S, T, \phi_1, \psi$:

$$U = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} U_\ell^m(r, \omega) Y_\ell^m(\theta, \varphi)$$

Déformation sphéroïdale

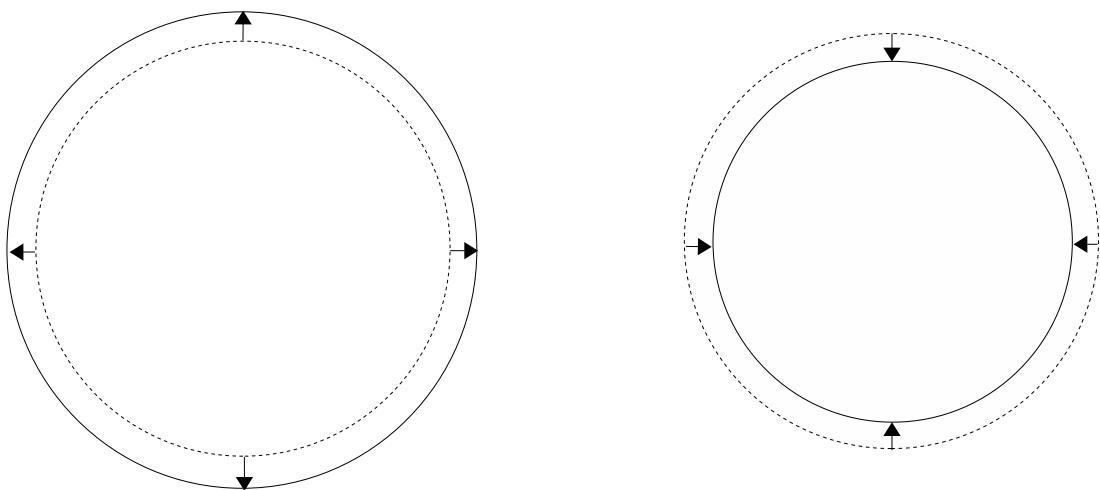
Exemple

$$\ell = 2, m = 0$$



Déformation radiale

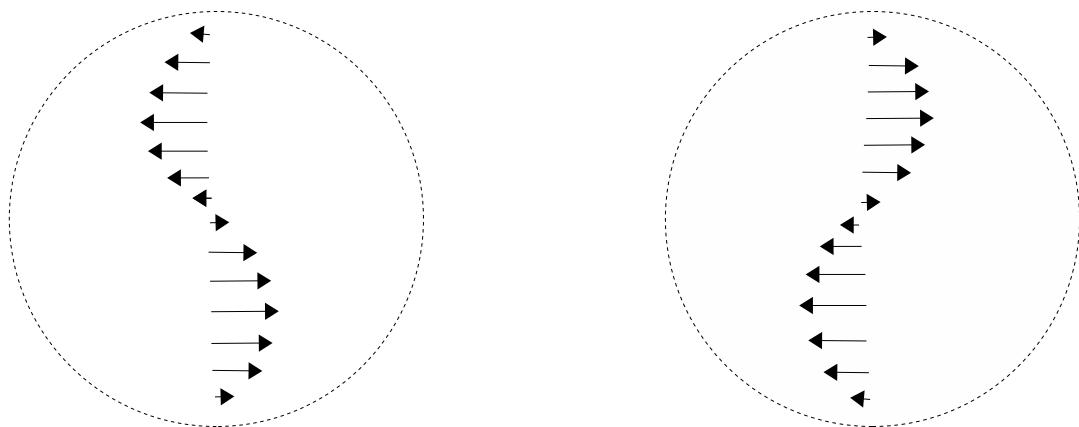
$$\ell = 0, m = 0$$



Déformation toroïdale

Exemple

$$\ell = 2, m = 0$$



Système sphéroïdal

Solide

$$\begin{aligned}
 \frac{dU}{dr} &= -\frac{2\lambda}{\beta r}U + \frac{1}{\beta}R + \frac{\ell(\ell+1)\lambda}{\beta r}V \\
 \frac{dR}{dr} &= \left[-\rho\omega^2 - \frac{4\rho g}{r} + \frac{4\mu(3\lambda+2\mu)}{\beta r^2} \right] U - \frac{4\mu}{\beta r}R \\
 &\quad + \ell(\ell+1) \left[\frac{\rho g}{r} - \frac{2\mu(3\lambda+2\mu)}{\beta r^2} \right] V \\
 &\quad + \frac{\ell(\ell+1)}{r}S + \rho\psi \\
 \frac{dV}{dr} &= \frac{1}{r}(V - U) + \frac{1}{\mu}S \\
 \frac{dS}{dr} &= \left[\frac{\rho g}{r} - \frac{2\mu(3\lambda+2\mu)}{\beta r^2} \right] U - \frac{\lambda}{\beta r}R \\
 &\quad + \left\{ -\rho\omega^2 + \frac{2\mu}{\beta r^2} [\lambda(2\ell^2+2\ell-1) \right. \\
 &\quad \left. + 2\mu(\ell^2+\ell-1)] \right\} V - \frac{3}{r}S + \frac{\rho}{r}\phi_1 \\
 \frac{d\phi_1}{dr} &= \psi - 4\pi G\rho U \\
 \frac{d\psi}{dr} &= \frac{4\pi G}{r}\rho\ell(\ell+1)V + \frac{\ell(\ell+1)}{r^2}\phi_1 - \frac{2}{r}\psi
 \end{aligned}$$

où

$$\beta = \lambda + 2\mu$$

Système sphéroïdal Fluide

$$\begin{aligned}\frac{dU}{dr} &= -\frac{2}{r}U + \frac{1}{\lambda}R + \frac{\ell(\ell+1)}{r}V \\ \frac{dR}{dr} &= \left(-\rho\omega^2 - \frac{4\rho g}{r}\right)U \\ &\quad + \ell(\ell+1) \left[\frac{\rho g}{r} - \frac{2\mu(3\lambda+2\mu)}{\beta r^2}\right]V + \rho\psi \\ \frac{d\phi_1}{dr} &= \psi - 4\pi G\rho U \\ \frac{d\psi}{dr} &= \frac{4\pi G}{r}\rho\ell(\ell+1)V + \frac{\ell(\ell+1)}{r^2}\phi_1 - \frac{2}{r}\psi \\ \rho\omega^2 r V &= \rho g U - R + \rho\phi_1\end{aligned}$$

Système radial

Solide

$$\frac{dU}{dr} = -\frac{2\lambda}{\beta r}U + \frac{1}{\beta}R$$
$$\frac{dR}{dr} = \left[-\rho\omega^2 - \frac{4}{r} \left(\rho g - \frac{\mu(3\lambda + 2\mu)}{\beta r} \right) \right] U - \frac{4\mu}{\beta r}R$$

Fluide

$$\frac{dU}{dr} = -\frac{2}{r}U + \frac{1}{\lambda}R$$
$$\frac{dR}{dr} = \left(-\rho\omega^2 - \frac{4}{r}\rho g \right) U$$

Système toroïdal

Solide

$$\frac{dW}{dr} = \frac{1}{r}W + \frac{1}{\mu}T$$

$$\frac{dT}{dr} = \left[-\rho\omega^2 + (\ell^2 + \ell - 2) \frac{\mu}{r^2} \right] W - \frac{3}{r}T$$

Conditions de continuité

Interface solide-fluide:

$$[U]_-^+ = 0$$

Interface solide-solide:

$$[U]_-^+ = [V]_-^+ = [W]_-^+ = 0$$

Partout:

$$[R]_-^+ = [S]_-^+ = [T]_-^+ = 0$$

$$[\phi_1]_-^+ = 0$$

$$[\psi]_-^+ = \left[\frac{d\phi_1}{dr} + 4\pi G \rho U \right]_-^+ = 0$$

Conditions à la surface

$$R(a) = \text{pression}$$

$$S(a) = \text{contrainte tangentielle}$$

$$T(a) = \text{contrainte tangentielle}$$

$$\psi(a) + \frac{\ell+1}{a} \phi_1(a) = (2\ell+1)a^{\ell-1} \phi_{\text{ext}\ell}$$

Résolution

- Analytique

(pour modèles simples: homogènes ou à couches homogènes → Exercices)

- Numérique

- Différences finies

$$y(x_0 + h) \simeq y(x_0) + h \frac{dy}{dx} \Big|_{x_0}$$

- Méthodes spectrales

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 - (a) **Oscillations libres
= Modes normaux**
 - (b) Déformation forcée
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Corde vibrante

- Corde de longueur L fixe à ses extrémités
- Elasticité linéaire
- Corde homogène: $\rho = \text{const}$, $T = \text{const}$
- Vibrations transversales

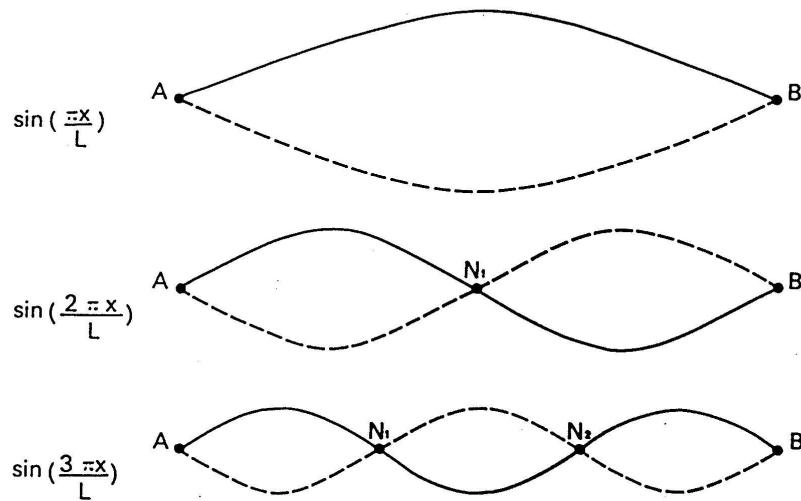
Fonctions propres

$$y_n = A \sin \frac{2\pi x}{\lambda_n} \cos \omega_n t$$

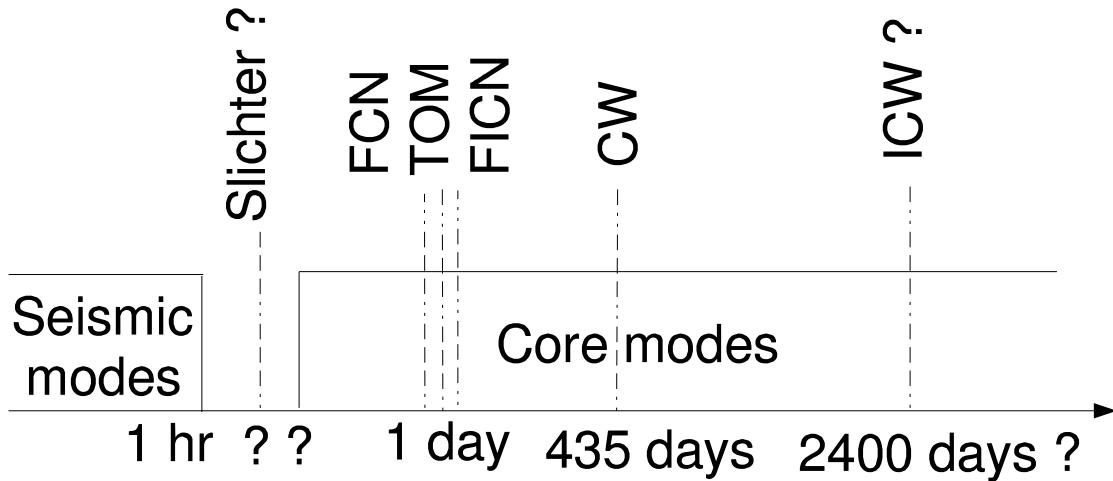
$$\lambda_n = \frac{2L}{n} \quad \text{where} \quad n = 1, \dots, \infty$$

Fréquences propres

$$\omega_n = \frac{n\pi}{L} \sqrt{\frac{T}{\rho}}$$

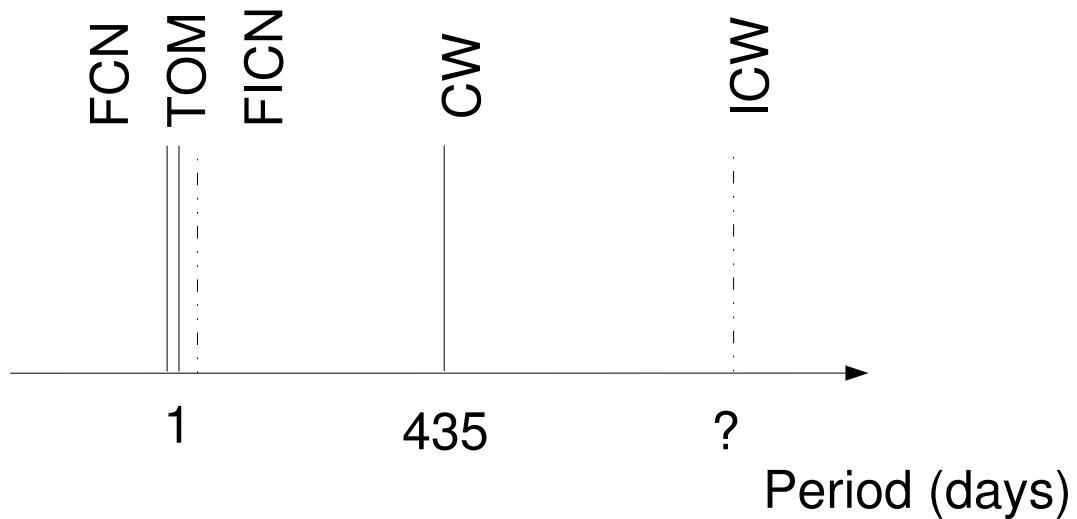


Modes normaux de la Terre



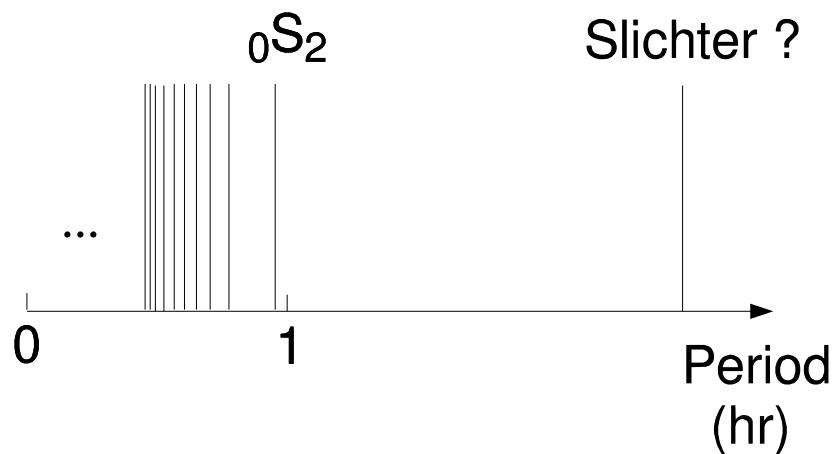
1. Modes sismiques (\rightarrow *Sismologie*)
2. Modes de rotation (\rightarrow *Géodésie*)
3. Modes de gravité-inertie du noyau liquide
(\rightarrow *Mécanique des Fluides*)

Modes de rotation



- i. Nutation libre du noyau liquide (FCN)
- ii. Nutation libre de la graine (FICN)
- iii. Mouvement de Chandler (CW)
- iv. Mouvement de Chandler de la graine (ICW)

Spectre des modes sismiques



- Forces de rappel: gravité + élasticité
(gravité pour $T \geq 100$ s)
- Observation de centaines de modes sismiques

Classification des modes sismiques

Sphéroïdaux nS_ℓ^m

- Variations de gravité
- Déplacements radiaux et tangentiels

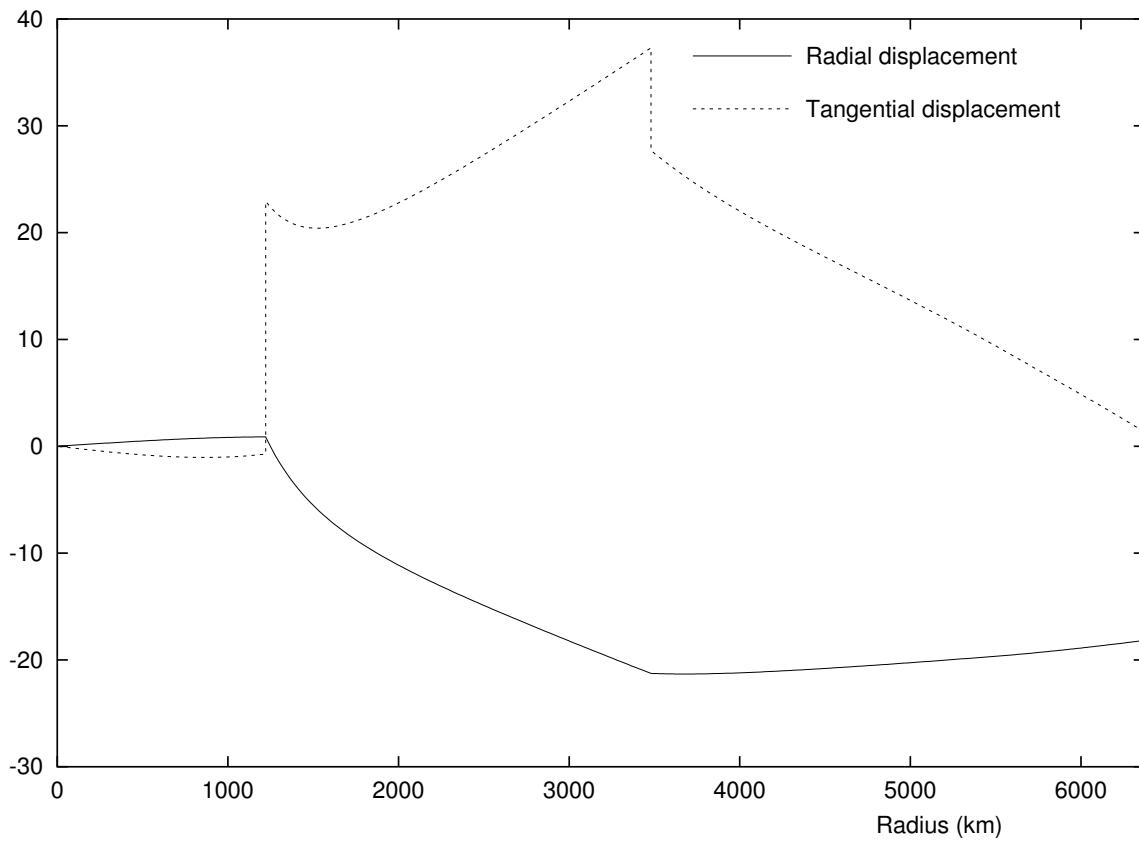
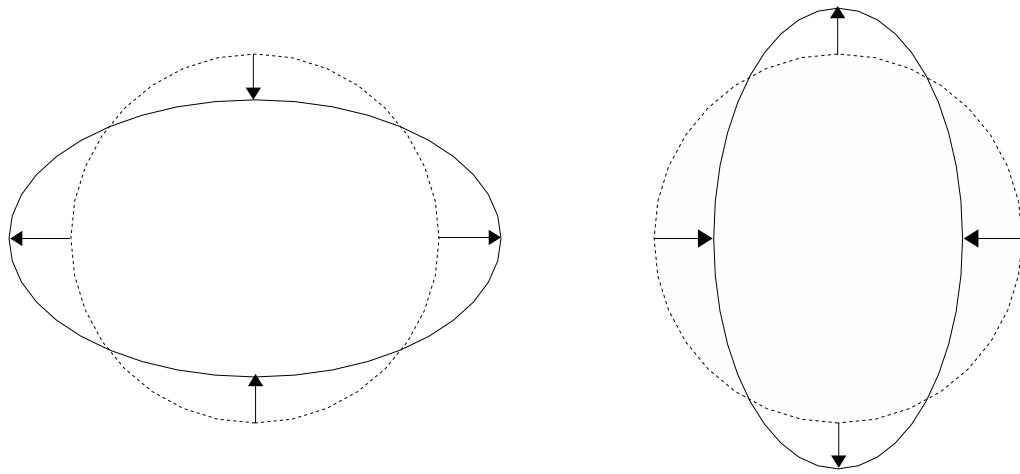
Toroïdaux nT_ℓ^m

- Pas de variation de gravité
- Déplacements tangentiels

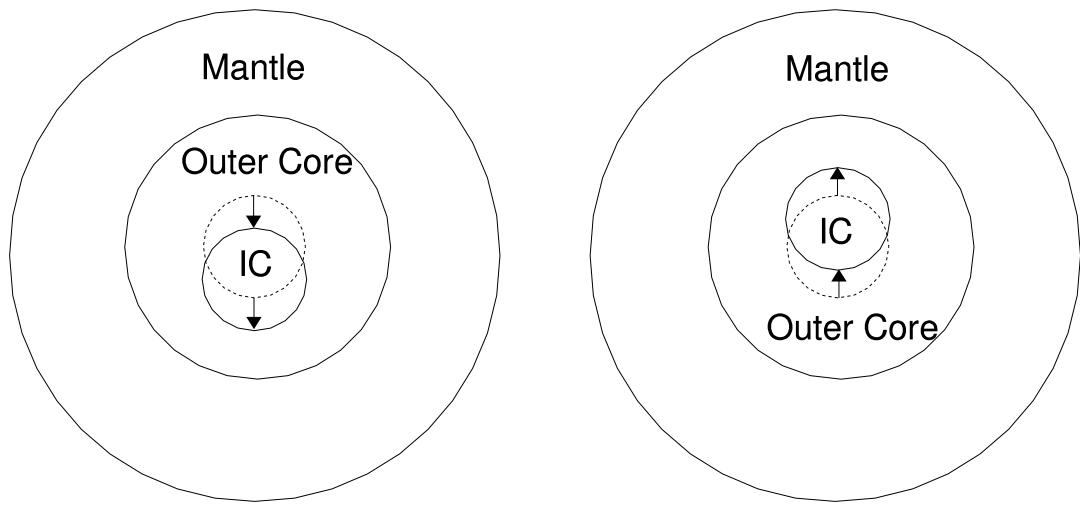
Nombres quantiques

- $n \geq 0$: ordre radial
- $\ell \geq 0$: degré angulaire
- $-l \leq m \leq l$: ordre angulaire

Dégénérescence
pas de dépendance en m



${}_0S_2 \simeq 54$ minutes



Mode de Slichter
 ${}_1S_1 \sim$ quelques heures

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Nombres de Love et Shida

(Force extérieure de volume)

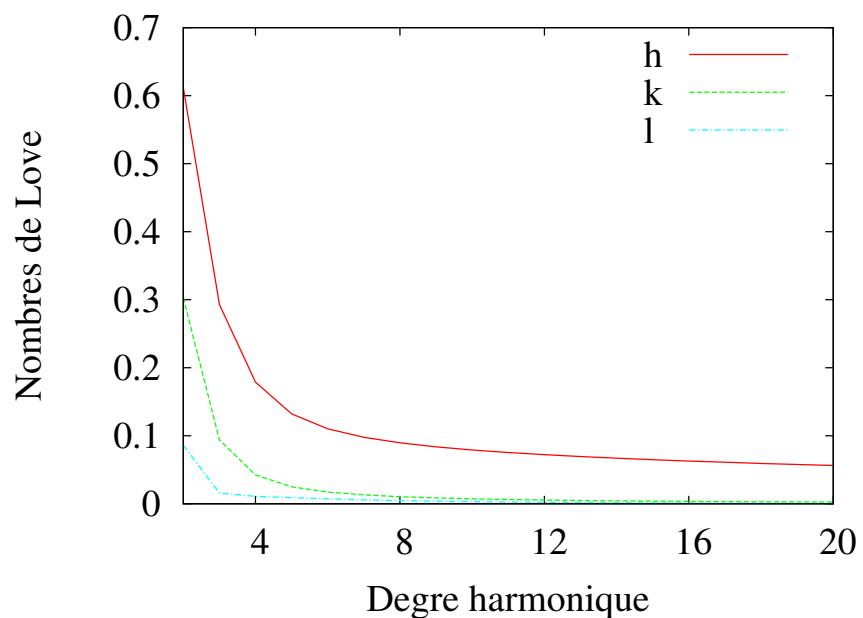
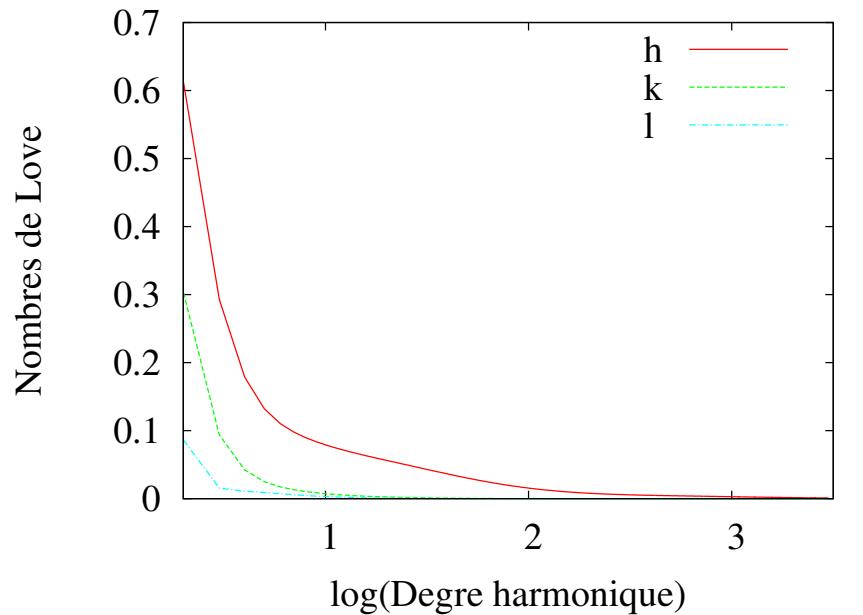
$$h_\ell = -\frac{U(a)}{\phi_{\text{ext}\ell}} g(a)$$

$$l_\ell = -\frac{V(a)}{\phi_{\text{ext}\ell}} g(a)$$

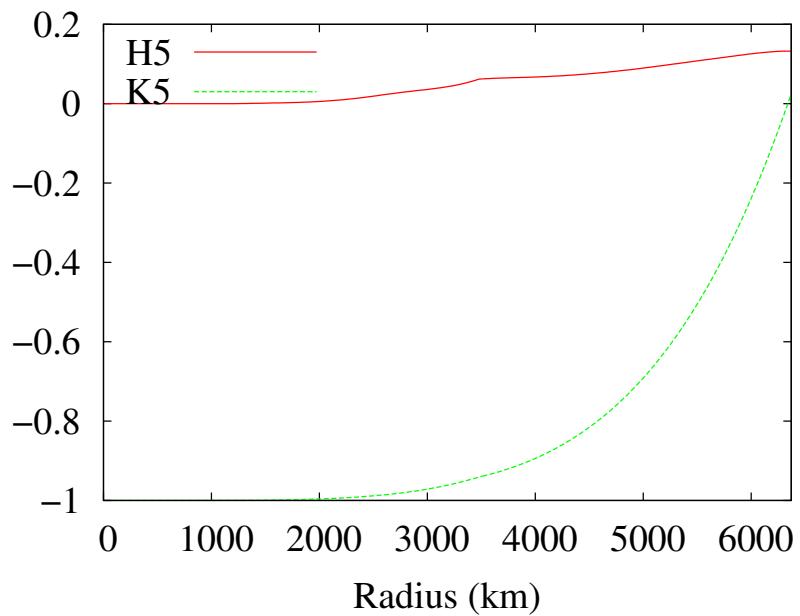
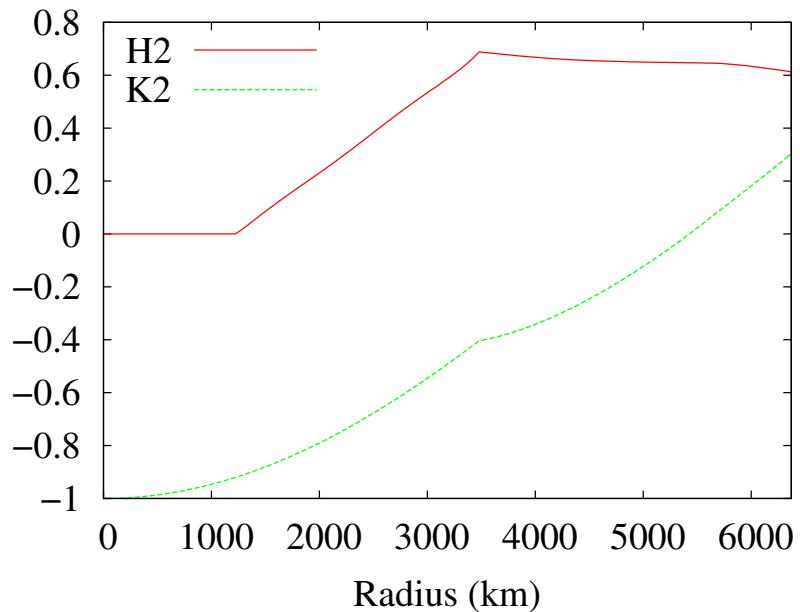
$$(1 + k_\ell) = \frac{\phi_1(a)}{\phi_{\text{ext}\ell}} = 1 + \frac{\Delta\phi(a)}{\phi_{\text{ext}\ell}}$$

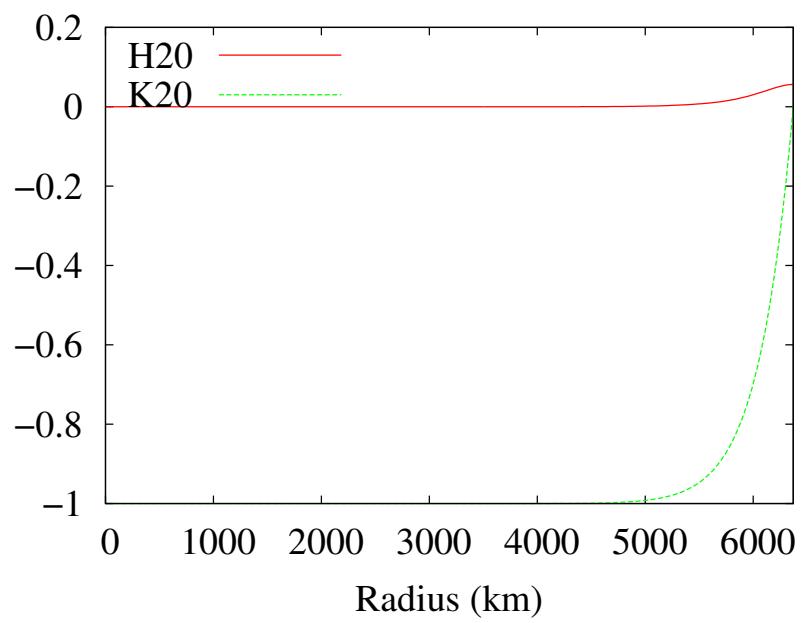
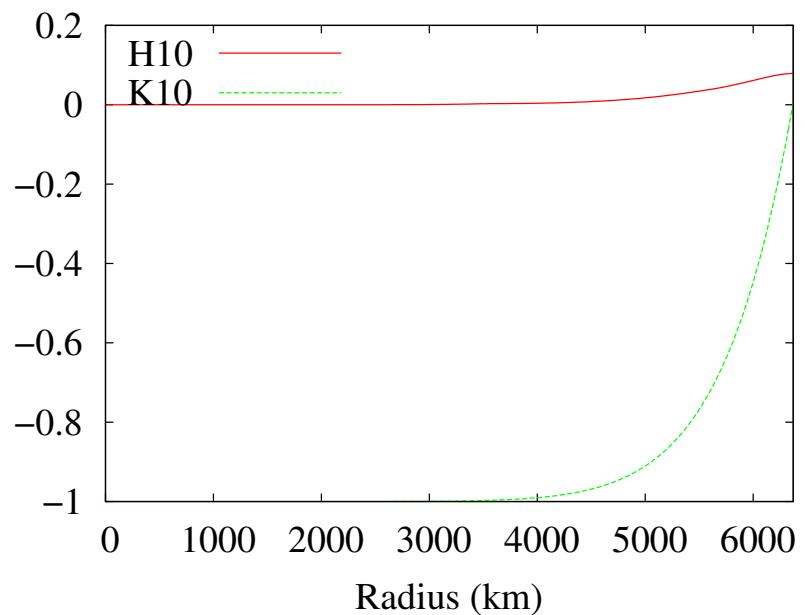
Nombres de Love

Modèle PREM



$H_\ell(r)$ et $K_\ell(r)$





Exemple k_2

Variation des composantes du tenseur d'inertie

$$\begin{pmatrix} A & I_{12} & I_{13} \\ I_{12} & B & I_{23} \\ I_{13} & I_{23} & C \end{pmatrix}$$

$$\begin{aligned} \phi^0 = -\frac{GM}{a} \left\{ \frac{a}{r} - \sum_{\ell=2}^{\infty} \left(\frac{a}{r} \right)^{\ell+1} [J_{\ell} P_{\ell}(\cos \theta) \right. \\ \left. + \sum_{m=1}^{\ell} C_{\ell}^m Y_{\ell}^{cm}(\theta, \varphi) + S_{\ell}^m Y_{\ell}^{sm}(\theta, \varphi)] \right\} \end{aligned}$$

$$J_2 = \frac{C - \frac{A+B}{2}}{Ma^2}$$

$$C_2^1 = -\sqrt{\frac{12\pi}{5}} \frac{I_{13}}{Ma^2} \quad S_2^1 = -\sqrt{\frac{12\pi}{5}} \frac{I_{23}}{Ma^2}$$

$$C_2^2 = \sqrt{\frac{12\pi}{5}} \frac{B - A}{2Ma^2} \quad S_2^2 = -\sqrt{\frac{12\pi}{5}} \frac{I_{12}}{Ma^2}$$

Soient:

- Modèle de Terre sphérique

$$\phi^0 = -\frac{GM}{r}$$

$$I^0 = \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix}$$

- Perturbation de degré 2, d'ordre 0 ou 1
 $\rightarrow \phi_{\text{ext}2}^m$ ($m = 0, 1$)

$$\Rightarrow \Delta\phi_2^m(a) = k_2 \phi_{\text{ext}2}^m(a)$$

$$\Rightarrow I = \begin{pmatrix} A + I_{11} & I_{12} & I_{13} \\ I_{12} & A + I_{22} & I_{23} \\ I_{13} & I_{23} & A + I_{33} \end{pmatrix}$$

Relations entre I_{ij} et $\phi_{\text{ext}2}^m(a)$?

On montre que (\rightarrow exercices)

- Pas de perturbation d'ordre 2 $\Rightarrow I_{11} = I_{22}$
- Linéarisation

$$I_{11} + I_{22} + I_{33} = 4 \int_{V^0} \rho^0 r U_2^m Y_2^m \, dV = 0$$
$$\Rightarrow I_{11} = I_{22} = -\frac{I_{33}}{2}$$

Conclusion

$$I_{31} = \sqrt{\frac{5}{12\pi}} \frac{k_2 a^3}{G} \phi_{\text{ext}2}^c{}^1(a)$$

$$I_{32} = \sqrt{\frac{5}{12\pi}} \frac{k_2 a^3}{G} \phi_{\text{ext}2}^s{}^1(a)$$

$$I_{33} = \frac{2}{3} \frac{k_2 a^3}{G} \phi_{\text{ext}2}^0(a)$$

Variation de la force centrifuge

$$\mathbf{f}_{\text{ext}} = -\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \boldsymbol{\omega}^0 \times (\boldsymbol{\omega}^0 \times \mathbf{r})$$

$$\boldsymbol{\omega}^0 = (0, 0, \Omega)$$

$$\boldsymbol{\omega} = (\omega_1, \omega_2, \Omega + \omega_3)$$

$$\begin{aligned}\Rightarrow \mathbf{f}_{\text{ext}} &= -\nabla \left\{ \Omega \left[\omega_1 xz + \omega_2 yz - \omega_3 (x^2 + y^2) \right] \right\} \\ &= \Omega \nabla \left[\sqrt{\frac{4\pi}{15}} (\omega_1 Y_2^{c1} + \omega_2 Y_2^{s1}) + \frac{2}{3} \omega_3 (1 - P_2) \right] r^2\end{aligned}$$

$$I_{31} = k_2 \frac{a^5}{3G} \Omega \omega_1$$

$$I_{32} = k_2 \frac{a^5}{3G} \Omega \omega_2$$

$$I_{33} = k_2 \frac{4a^5}{9G} \Omega \omega_3$$

Observable Facteur gravimétrique

$$\delta_\ell = 1 + \frac{2}{\ell} h_\ell - \frac{\ell+1}{\ell} k_\ell$$

$\Delta g \mathbf{e}_r$ mesuré par un gravimètre en $r = a$:

1. force de marée

$$-\frac{d\phi_{\text{ext}\ell}}{dr} \mathbf{e}_r = -\frac{\ell}{r} \phi_{\text{ext}\ell} \mathbf{e}_r$$

2. déplacement radial du gravimètre

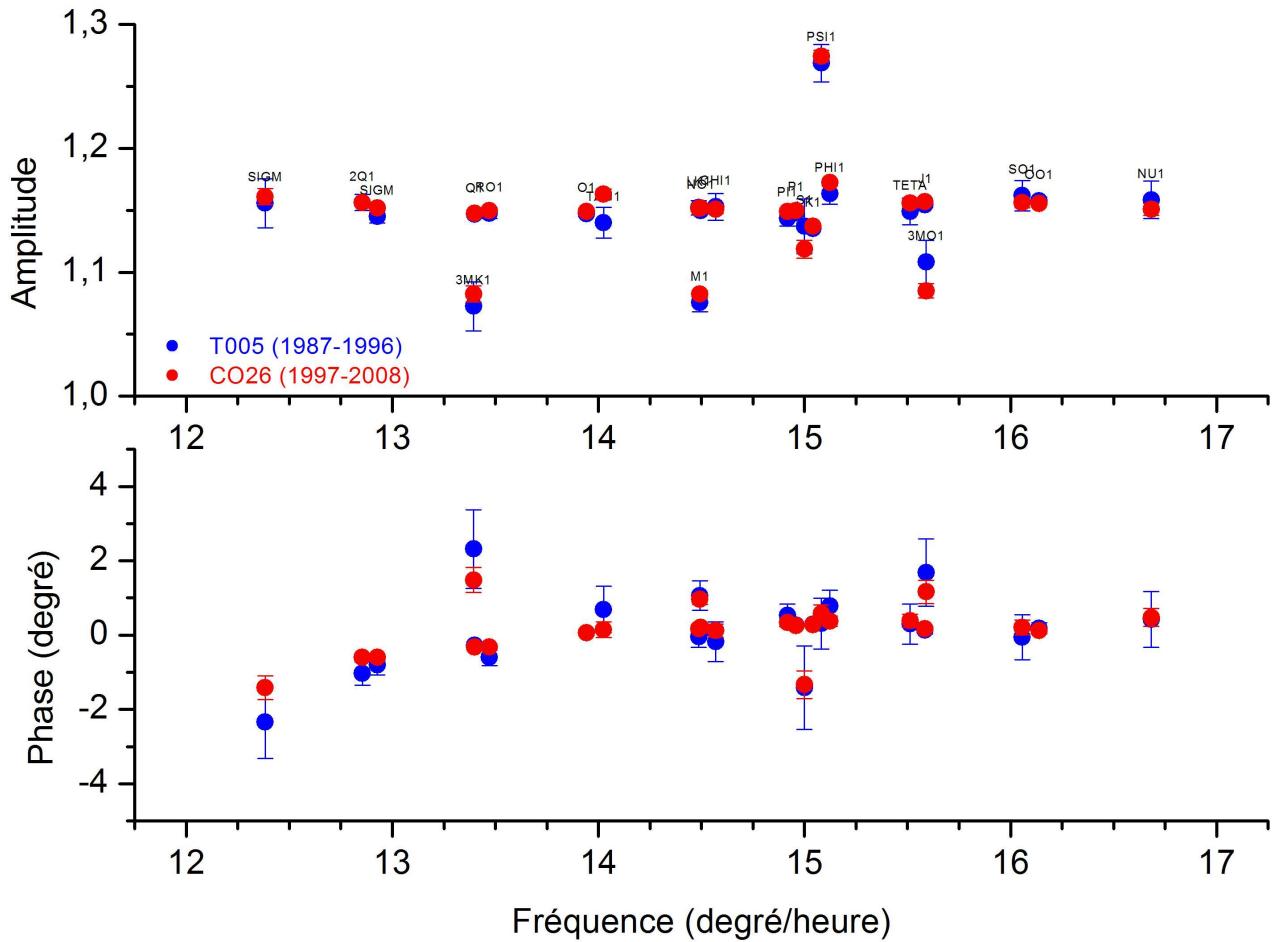
$$(\mathbf{s} \cdot \nabla) \mathbf{g} = -U \frac{dg}{dr} \mathbf{e}_r = \frac{2gU}{r} \mathbf{e}_r = -\frac{2h_\ell}{r} \phi_{\text{ext}\ell} \mathbf{e}_r$$

3. redistribution des masses en surface

$$\frac{\ell+1}{a} k_\ell \phi_{\text{ext}\ell} \mathbf{e}_r$$

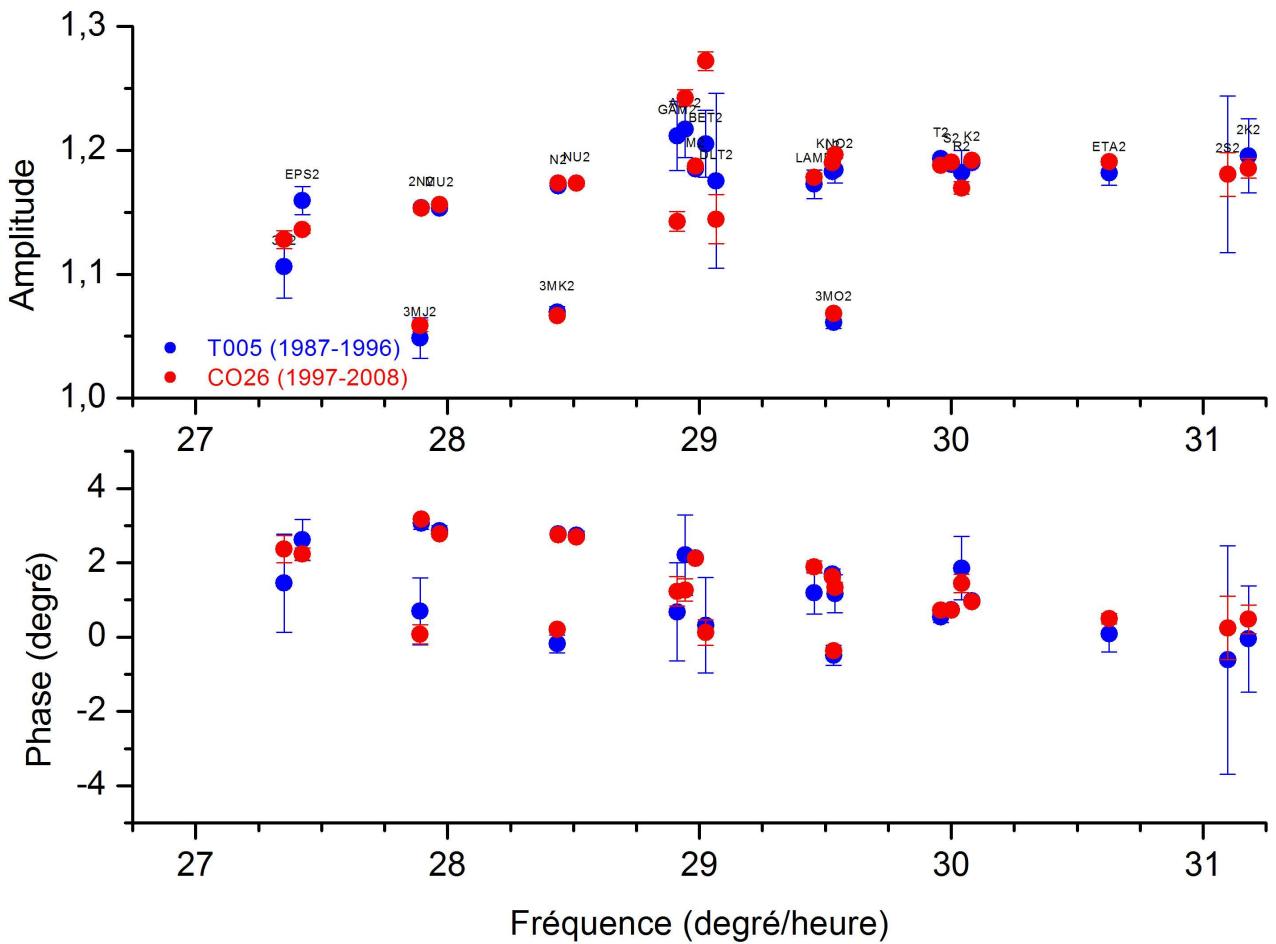
Observations

Diurne

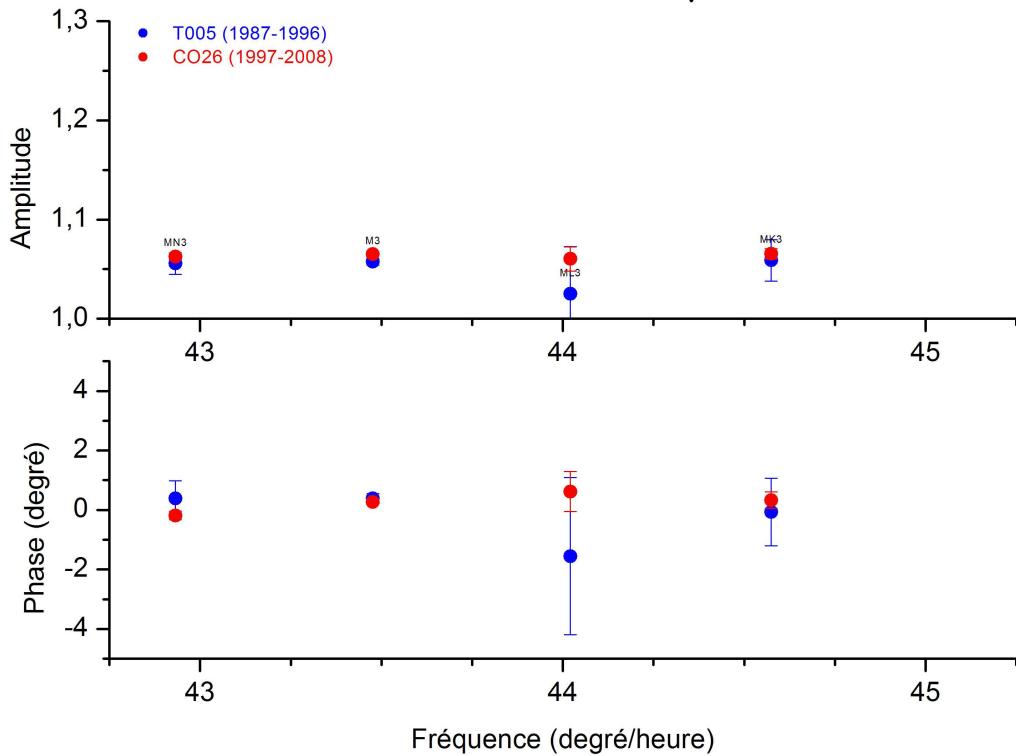


(Gravimètres supraconducteurs T005 puis
C026, Strasbourg. J.-P. Boy)

Semi-diurne



Ter-diurne



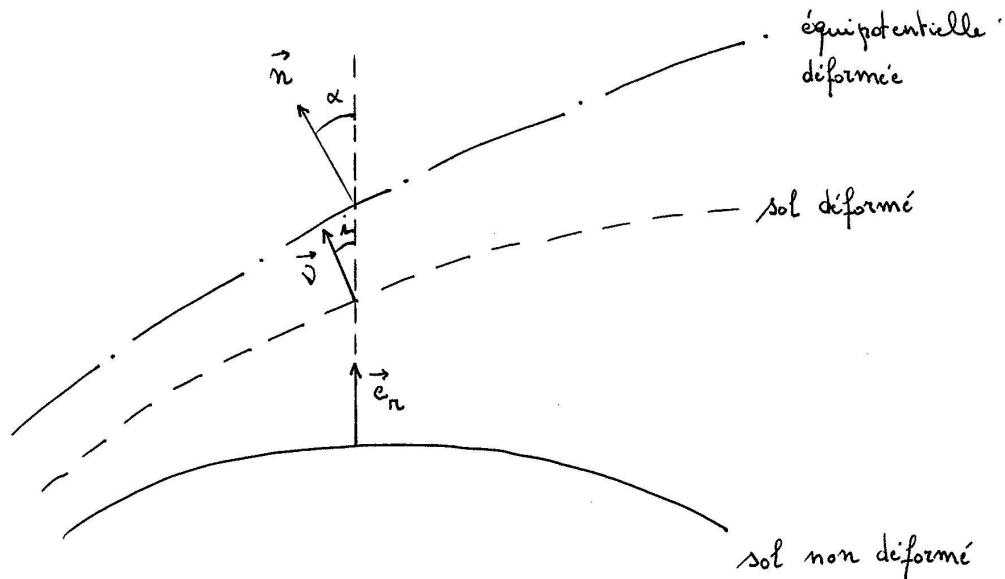
Analyse de marées

Melchior P. (1983): *The tides of the planet Earth*, Pergamon Press.

Zürn W. (1997): Earth tide observations and interpretation, in: *Lecture Notes in Earth Sciences - Tidal phenomena*, eds Wilhem, Zürn, Wenzel, Springer.

Observable: Facteur clinométrique

$$\gamma_\ell = 1 + k_\ell - h_\ell$$



- $\alpha - i$ mesuré par un inclinomètre
 $\simeq \sin(\alpha - i) = |\mathbf{n} \times \boldsymbol{\nu}| \simeq |\mathbf{n} \times \mathbf{e}_r| - |\mathbf{e}_r \times \boldsymbol{\nu}|$
- \mathbf{n} normale à l'équipotentielle
 $r = a + (1 + k_\ell) \frac{\phi_{\text{ext}\ell}}{g}$
- $\boldsymbol{\nu}$ normale au sol déformé
 $r = a + h_\ell \frac{\phi_{\text{ext}\ell}}{g}$

Observable: Facteur extensométrique

$$\sigma_\ell = 2h_\ell - \ell(\ell + 1)l_\ell$$

Nombres de Love en pression

(Pression en surface p)

$$\bar{h}_\ell = \frac{U(a)}{p_\ell} \rho g$$

$$\bar{l}_\ell = \frac{V(a)}{p_\ell} \rho g$$

$$\bar{k}_\ell = \frac{\Delta\phi(a)}{p_\ell} \rho$$

Nombres de Love en surcharge

(Surcharge superficielle à symétrie axiale σ)

1. attraction directe

$$\phi_{\text{ext}\ell} = \frac{3g}{2\ell+1} \left(\frac{r}{a}\right)^\ell \frac{\sigma_\ell}{\bar{\rho}}$$

2. pression en surface

$$p = \sigma g$$

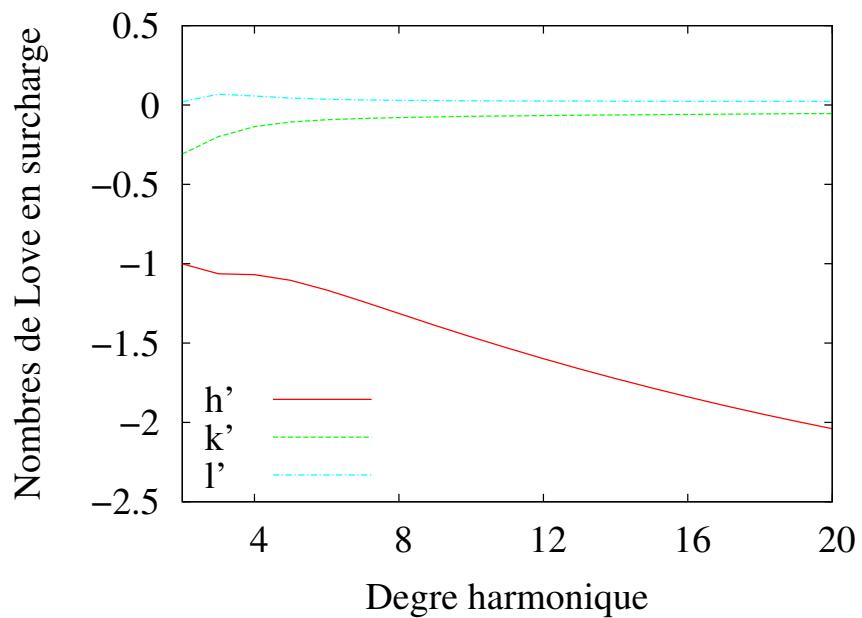
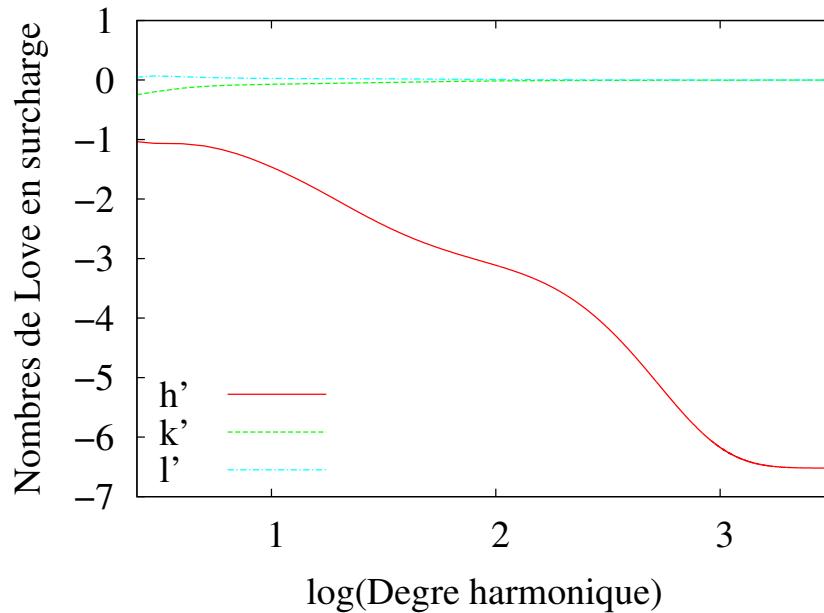
$$h'_\ell = -\frac{(2\ell + 1)U(a)}{3\sigma_\ell} \bar{\rho}$$

$$l'_\ell = -\frac{(2\ell + 1)V(a)}{3\sigma_\ell} \bar{\rho}$$

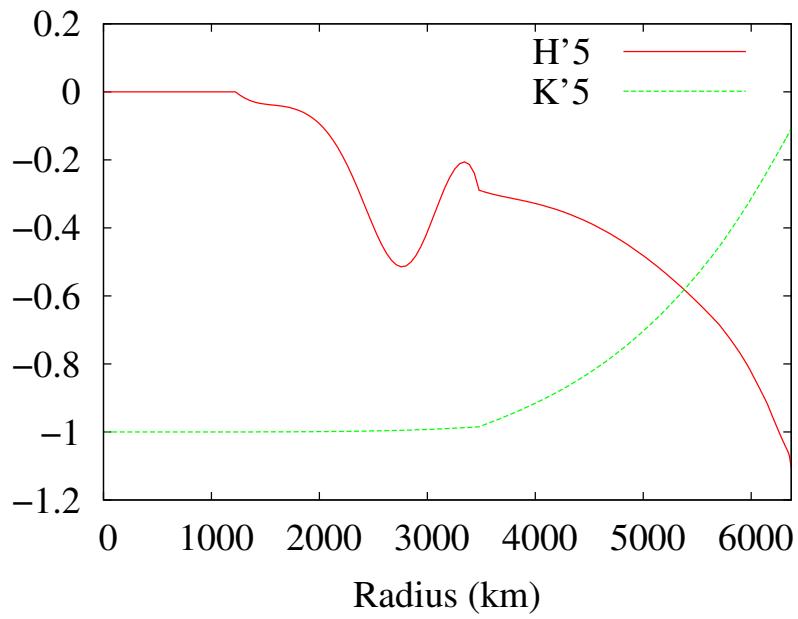
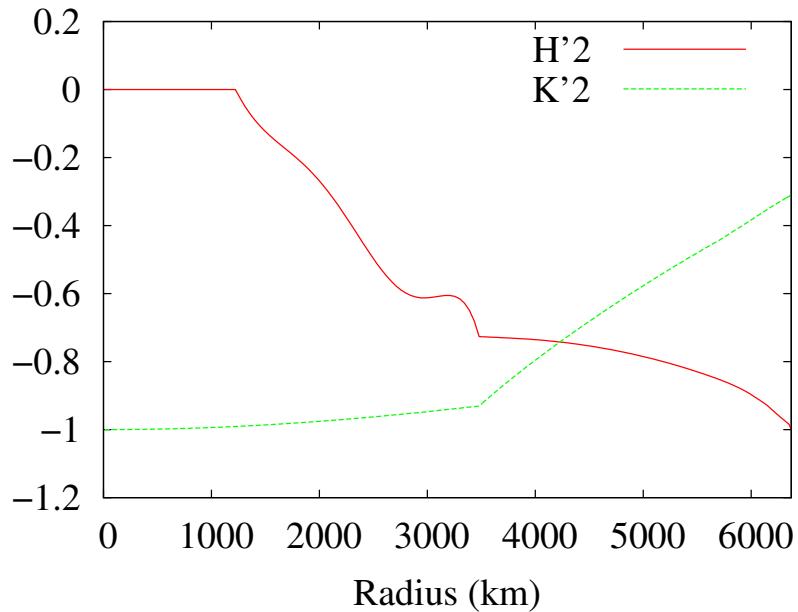
$$(1 + k'_\ell) = \frac{(2\ell + 1)\phi_1(a)}{3\sigma_\ell} \frac{\bar{\rho}}{g}$$

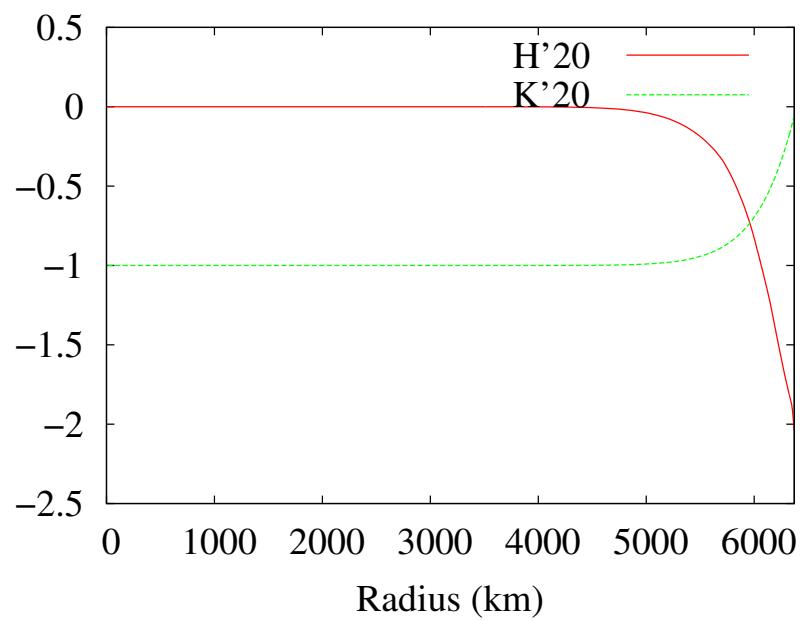
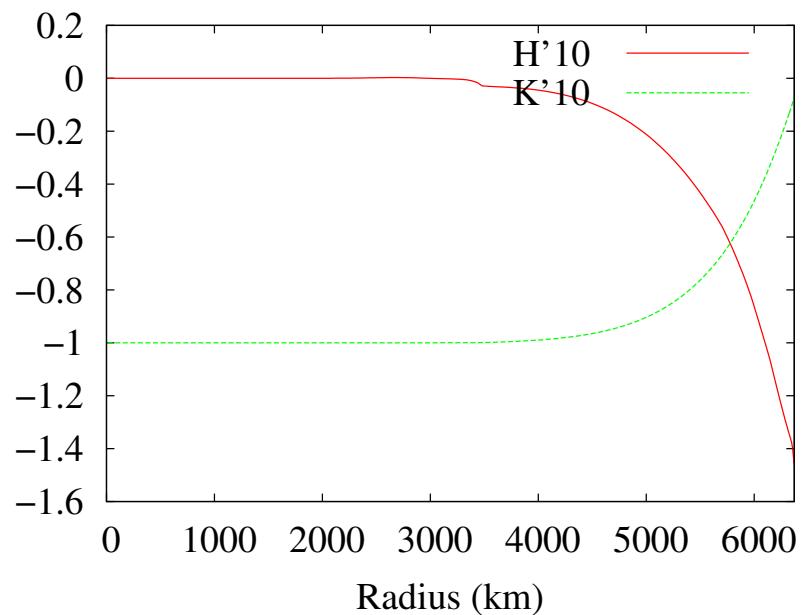
Nombres de Love en surcharge

Modèle PREM



$H'_\ell(r)$ et $K'_\ell(r)$





Fonctions de Green pour une surcharge

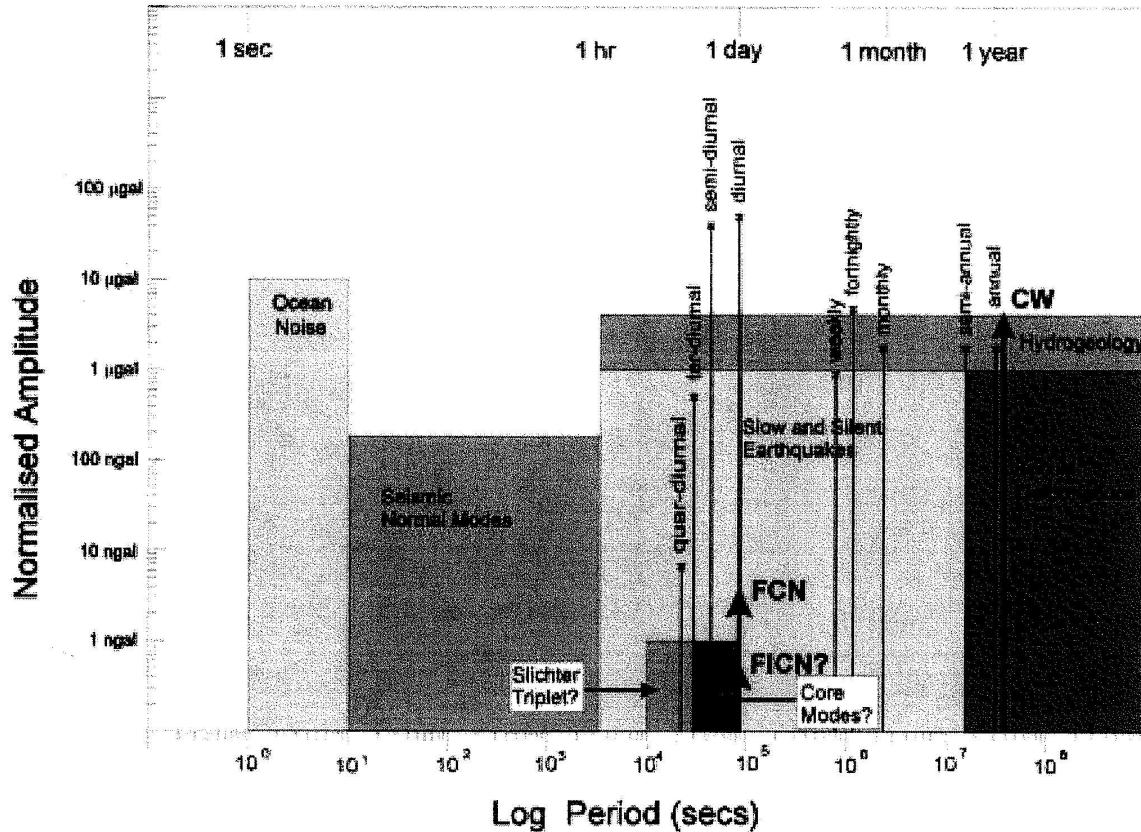
Déformation due à une masse unitaire ponctuelle en surface?

→ évaluation de séries du type

$$\sum_{\ell=0}^{\infty} h_{\ell} P_{\ell}(\cos \theta)$$

Farrell W. E. (1972): Deformation of the Earth by surface loads, *Rev. Geophys. Space Phys.*, **10**, 761–797.

1. Rappels de mécanique des milieux continus
2. Déformation d'une Terre élastique
 - (a) Oscillations libres
 - (b) Déformation forcée
3. **Déformation d'une Terre visco-élastique**



- Bande sismique: réponse (an)élastique
- Rebond post-glaciaire (~ 10000 ans): réponse viscoélastique
- Entre les deux:
Burger généralisé? (Yuen et Peltier 1982)

Rhéologie linéaire

Elasticité linéaire et isotrope
(loi de Hooke)

$$\begin{aligned}\delta t &= \lambda^0 (\nabla \cdot s) I + 2\mu^0 e \\ &= \kappa^0 (\nabla \cdot s) I + 2\mu^0 d \\ \kappa^0 &= \lambda^0 + 2/3\mu^0\end{aligned}$$

$$d = e - \frac{1}{3}(\nabla \cdot s)I$$

Fluide newtonien

$$\begin{aligned}\delta t &= \lambda'^0 \frac{\partial}{\partial t} (\nabla \cdot s) I + 2\nu^0 \frac{\partial}{\partial t} e \\ &= \kappa'^0 \frac{\partial}{\partial t} (\nabla \cdot s) I + 2\nu^0 \frac{\partial}{\partial t} d\end{aligned}$$

ν^0 = viscosité dynamique

Hypothèse fréquente: $\kappa'^0 = 0$

Anélasticité linéaire isotrope

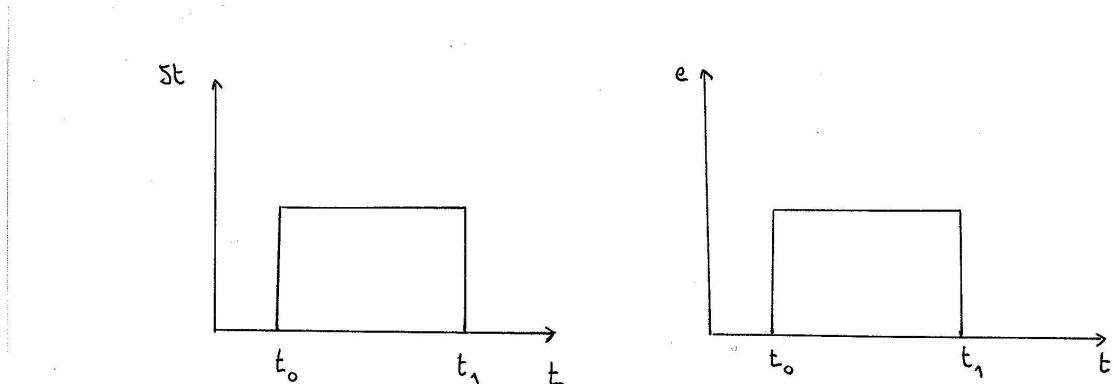
$$\begin{aligned}\delta t &= \int_{-\infty}^t \kappa^0(t-t') \frac{\partial}{\partial t'} (\nabla \cdot s) dt' I \\ &\quad + \int_{-\infty}^t 2\mu^0(t-t') \frac{\partial}{\partial t'} d dt'\end{aligned}$$

Analogies 1-D

“Ressort” et “piston”

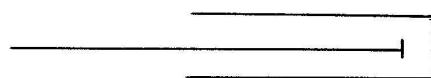
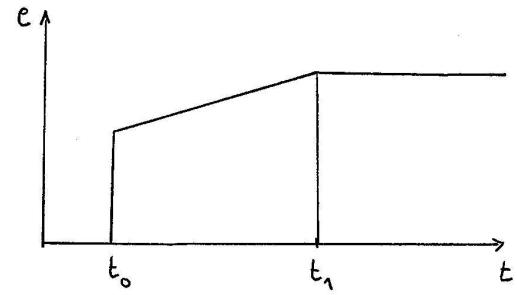
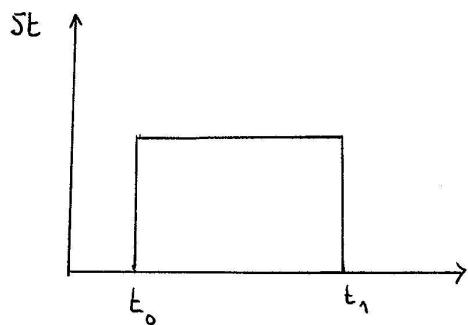
Elasticité linéaire

$$\delta t = 2\mu e$$



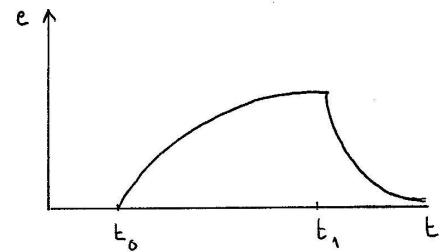
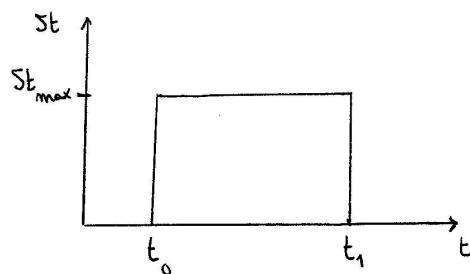
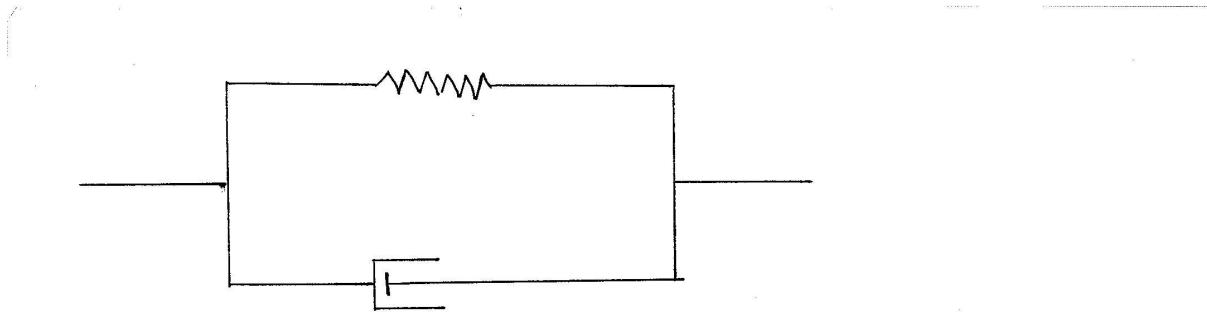
Viscosité linéaire

$$\delta t = 2\nu \frac{\partial}{\partial t} e$$



Kelvin-Voigt

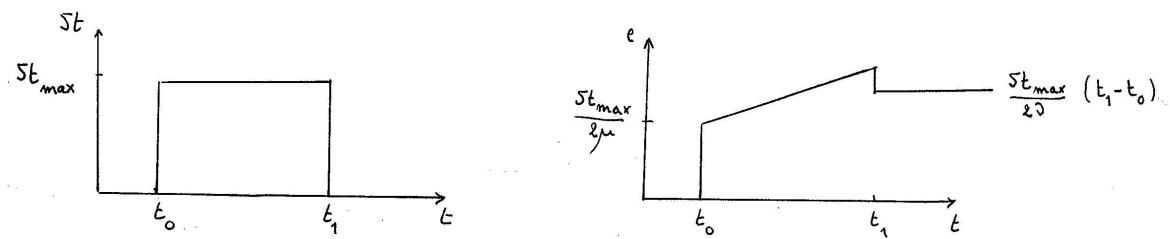
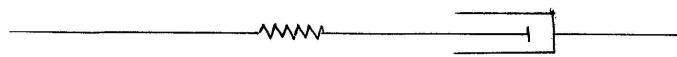
$$\delta t = 2\mu e + 2\nu \frac{\partial}{\partial t} e$$



$$\begin{aligned}
 \delta e &= \frac{(\delta t)_{\max}}{2\mu} \left[1 - e^{-\frac{\mu}{\nu}(t-t_0)} \right] & t_0 < t < t_1 \\
 &= \frac{(\delta t)_{\max}}{2\mu} e^{-\frac{\mu}{\nu}t} & t > t_1 \quad (\text{si } t_1 \gg \frac{\mu}{\nu})
 \end{aligned}$$

Maxwell

$$\frac{\partial}{\partial t}e = \frac{1}{2\mu}\frac{\partial}{\partial t}\delta t + \frac{1}{2\eta}\delta t$$



$$\begin{aligned}\delta e &= \frac{(\delta t)_{max}}{2\mu} + \frac{(\delta t)_{max}}{2\nu}(t - t_0) \quad t_0 < t < t_1 \\ &= \frac{(\delta t)_{max}}{2\nu}(t_1 - t_0) \quad t > t_1\end{aligned}$$

Anélasticité dans la bande sismique

$$\delta t = \kappa^0(\omega)[1 + iQ_\kappa^{-1}(\omega)](\nabla \cdot \mathbf{s})\mathbf{I} + 2\mu^0(\omega)[1 + iQ_\mu^{-1}(\omega)]\mathbf{d}$$

$$Q_\kappa \simeq \text{const}$$

$$Q_\mu \simeq \text{const}$$

$$\frac{\kappa^0(\omega)}{\kappa^0(\omega_{\text{ref}})} \simeq 1 + \frac{2}{\pi Q_\kappa} \ln \frac{\omega}{\omega_{\text{ref}}}$$

$$\frac{\mu^0(\omega)}{\mu^0(\omega_{\text{ref}})} \simeq 1 + \frac{2}{\pi Q_\mu} \ln \frac{\omega}{\omega_{\text{ref}}}$$

Viscoélasticité

Equation constitutive: rhéologie de Maxwell

$$\begin{aligned} \frac{d}{dt} \underline{\delta t}_{ij} + \frac{\mu}{\nu} (\underline{\delta t}_{ij} - \frac{1}{3} \delta_{ij} \sum_k \underline{\delta t}_{kk}) \\ = 2\mu \frac{d}{dt} \underline{e}_{ij} + \lambda \delta_{ij} \sum_k \frac{d}{dt} \underline{e}_{kk} \\ \nu = \text{viscosité} \end{aligned}$$

$\xrightarrow{\text{T.F.}}$ Principe de correspondance

$$\delta t_{ij} = \tilde{\lambda} \delta_{ij} \sum_k e_{kk} + 2\tilde{\mu} e_{ij}$$

$$\tilde{\lambda} = \frac{i\omega\lambda + \frac{\mu}{\nu}(\lambda + \frac{2\mu}{3})}{i\omega + \frac{\mu}{\nu}}$$

$$\tilde{\mu} = \frac{i\omega\mu}{i\omega + \frac{\mu}{\nu}}$$

Nombres de Love viscoélastiques

= réponse en surface à une surcharge impulsive [en $\delta(t)$]

$\lim_{\omega \rightarrow \infty} \tilde{h}'_\ell(\omega) = h'_\ell$: limite élastique

$\lim_{\omega \rightarrow 0} \tilde{h}'_\ell(\omega) = h'^f_\ell$: limite fluide

$$\tilde{h}'_\ell(t) = \mathcal{T}\mathcal{F}^{-1}[\tilde{h}'_\ell(\omega)]$$

Même chose pour $\tilde{k}'_\ell(\omega)$ et $\underline{\tilde{k}}'_\ell(t)$

Exemple

Modèle de viscosité
 $\nu = 10^{22}$ Pa s dans le manteau

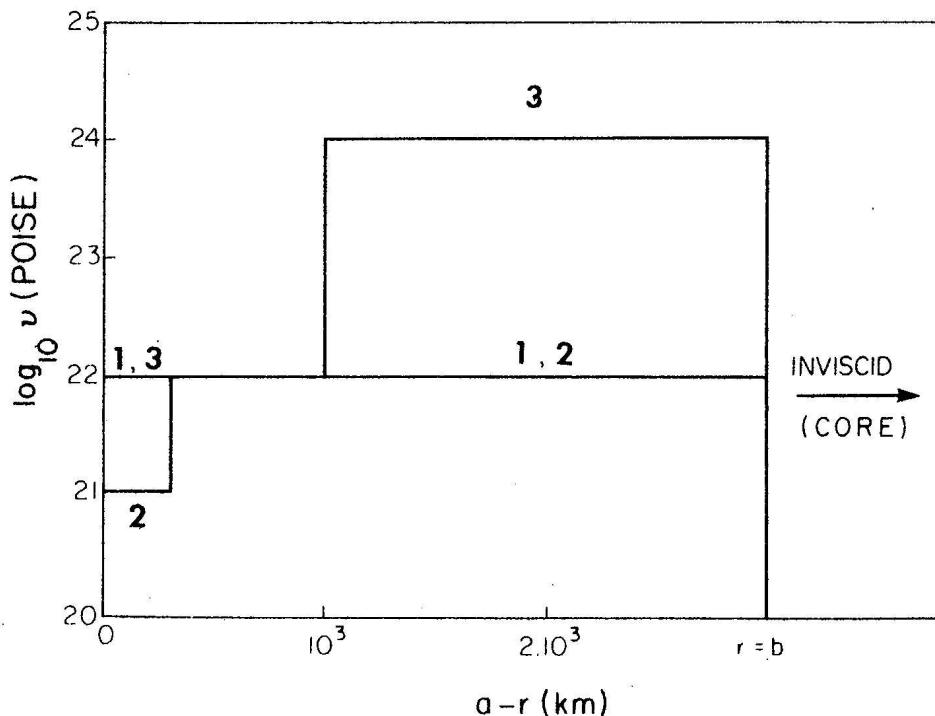


Fig. 1. Mantle viscosity models 1–3, described in text.

Peltier W. R. (1974): The impulse response of a Maxwell Earth, *Rev. Geoph. and Space Ph.*, **12**, 649–669.

Nombres de Love viscoélastiques

$$h'_\ell(\omega)$$

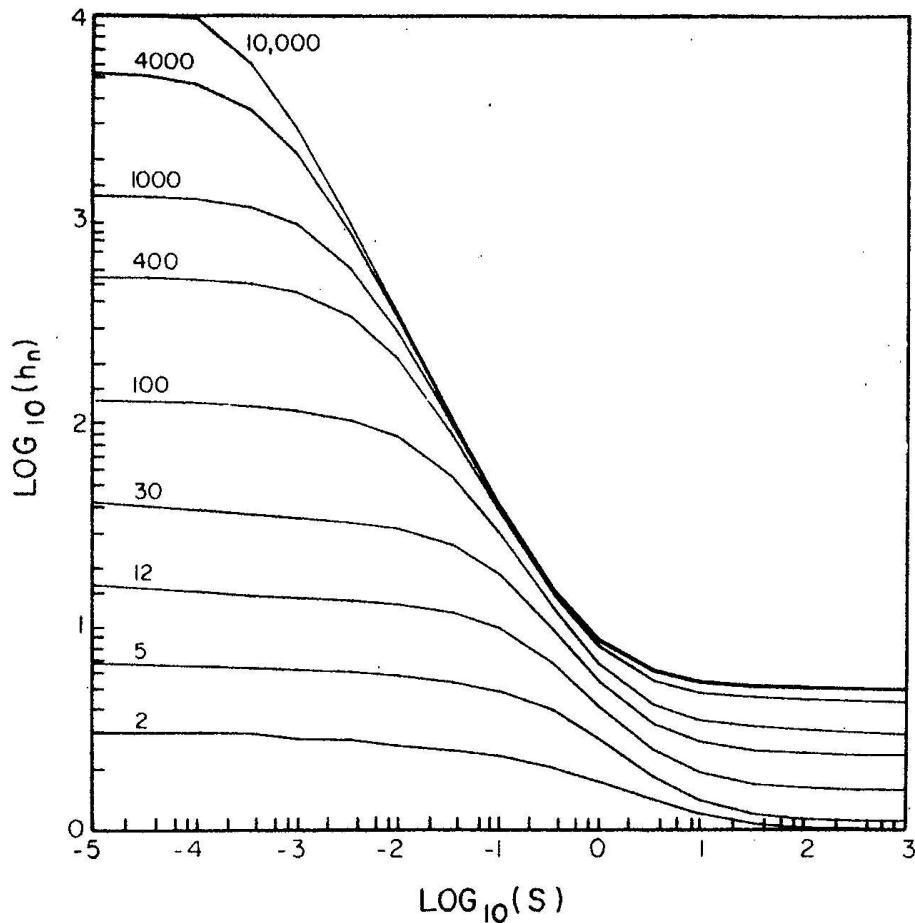


Fig. 2. Love number s spectra for viscosity model 1 (h_n). The corresponding value of n is marked beside each curve. Note the asymptotes for large and small s , which correspond to the elastic and sostatic limits, respectively.

$$\ell k'_\ell(\omega)$$

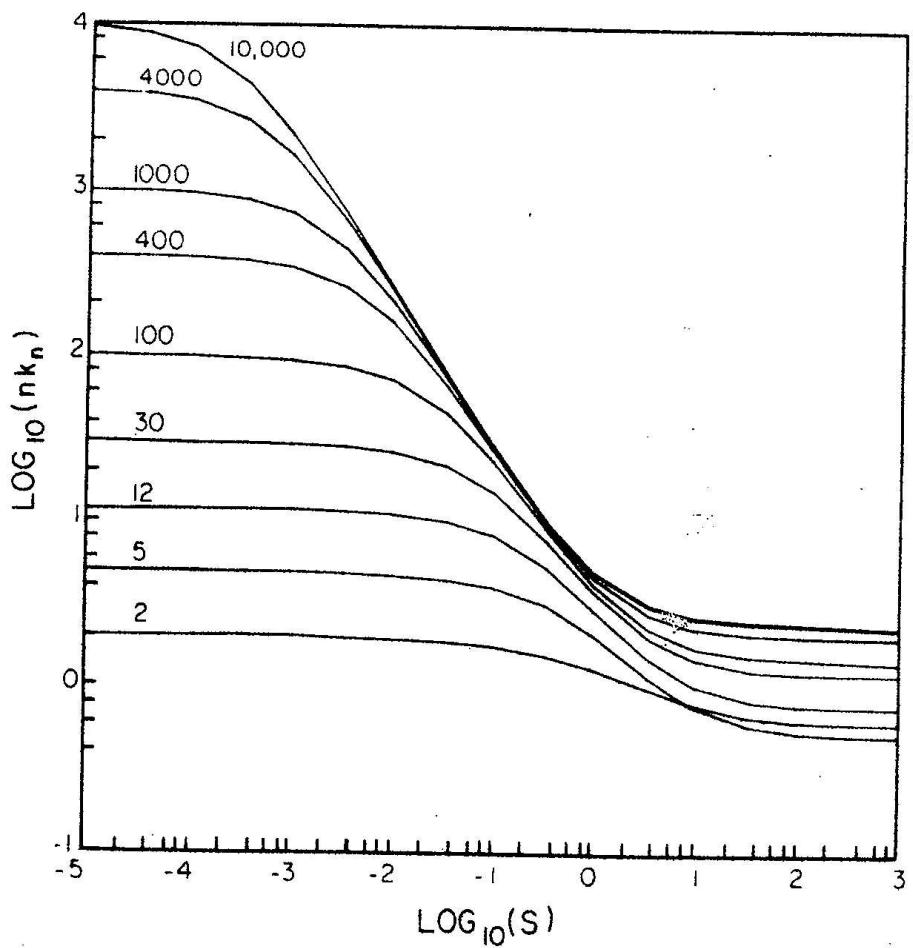


Fig. 3. Love number s spectra for viscosity model 1 (nk_n). Note the intersection of the small n spectra at intermediate s values.

$$\underline{h}'_n(t)$$

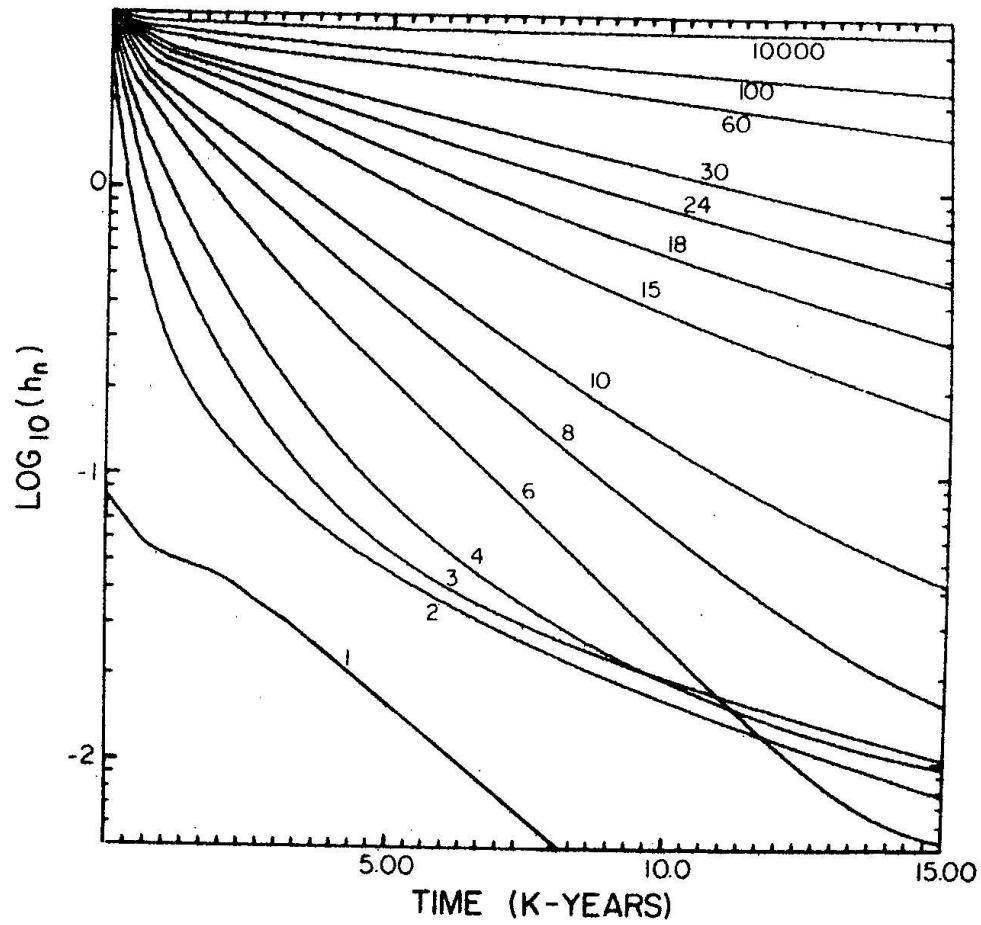
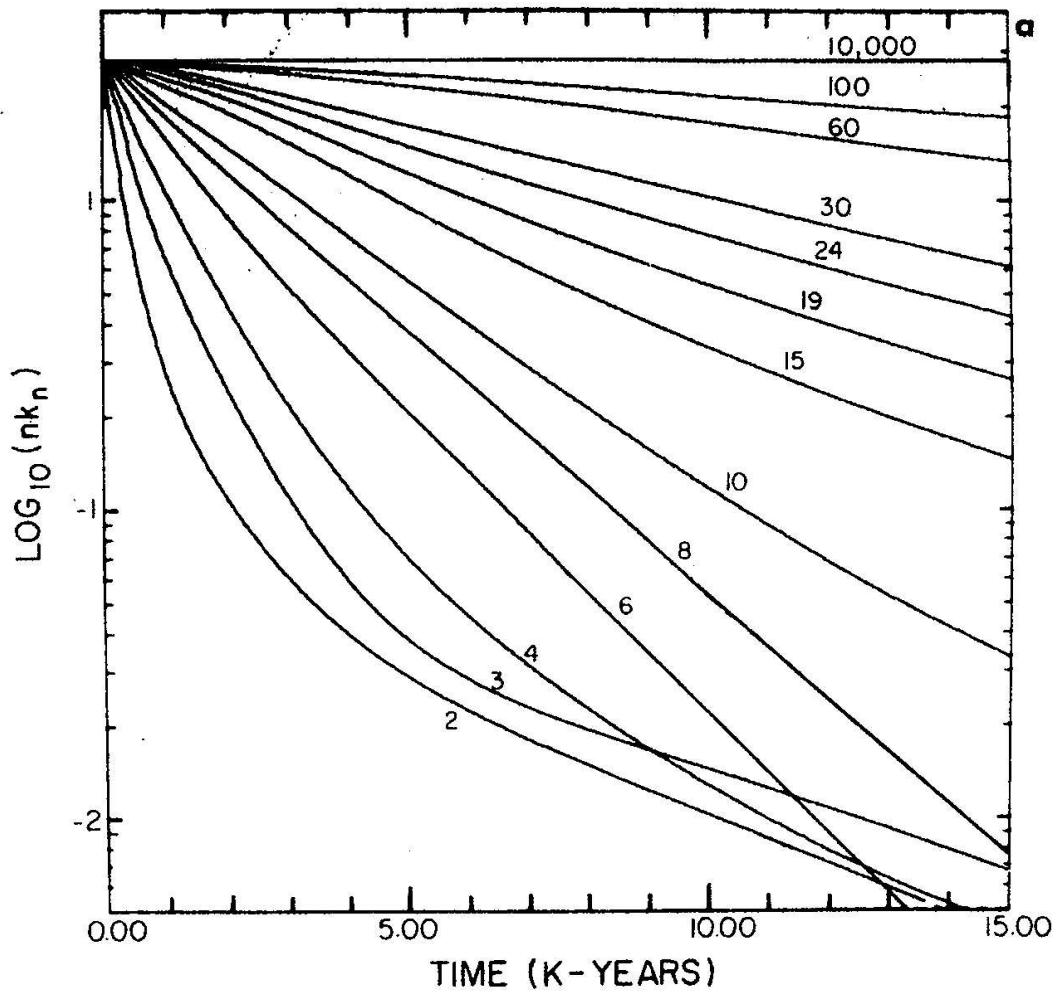


Fig. 4. Time dependent Love numbers h_n for model I. The plot is log linear. Note the strongly nonexponential style of relaxation of the low-order Love numbers and the extremely slow exponential style of decay of the $n = 10,000$ Love number.

$$\ell k'_{\ell}(t)$$



Déplacement radial

$$U_\ell(a, \omega) = \frac{1}{g} h'_\ell(\omega) \phi_{\text{ext}\ell}(a, \omega)$$

$$\xrightarrow{\text{T.F.}} \underline{U}_\ell(a, t) = \frac{1}{g} \underline{h}'_\ell(t) \star \underline{\phi}_{\text{ext}\ell}(a, t)$$

Produit dans l'espace fréquentiel

$\xrightarrow{\text{T.F.}}$

produit de convolution dans l'espace temporel

Temps de relaxation (\rightarrow exercices)

Modèle homogène incompressible

$\rho = \text{const}$, $\mu = \text{const}$, $\lambda \rightarrow \infty$, $\nu = \text{const}$

h'_ℓ élastique:

$$h'_\ell = -\frac{(2\ell + 1)}{3} \frac{1}{1 + \mu_\ell}$$

$$\mu_\ell = \frac{2\ell^2 + 4\ell + 3}{\ell} \frac{\mu}{\rho g a}$$

P.C. $\Rightarrow h'_\ell(\omega)$ viscoélastique:

$$h'_\ell(\omega) = h'_\ell \left(1 + \frac{\mu_\ell}{\tau_\ell} \frac{1}{i\omega + \frac{1}{\tau_\ell}} \right)$$

$$\tau_\ell = (1 + \mu_\ell) \frac{\nu}{\mu}$$

T.F. $\tilde{h}'_\ell(t) = h'_\ell \left[\delta(t) + \frac{\mu_\ell}{\tau_\ell} e^{-t/\tau_\ell} H(t) \right]$

Temps de relaxation

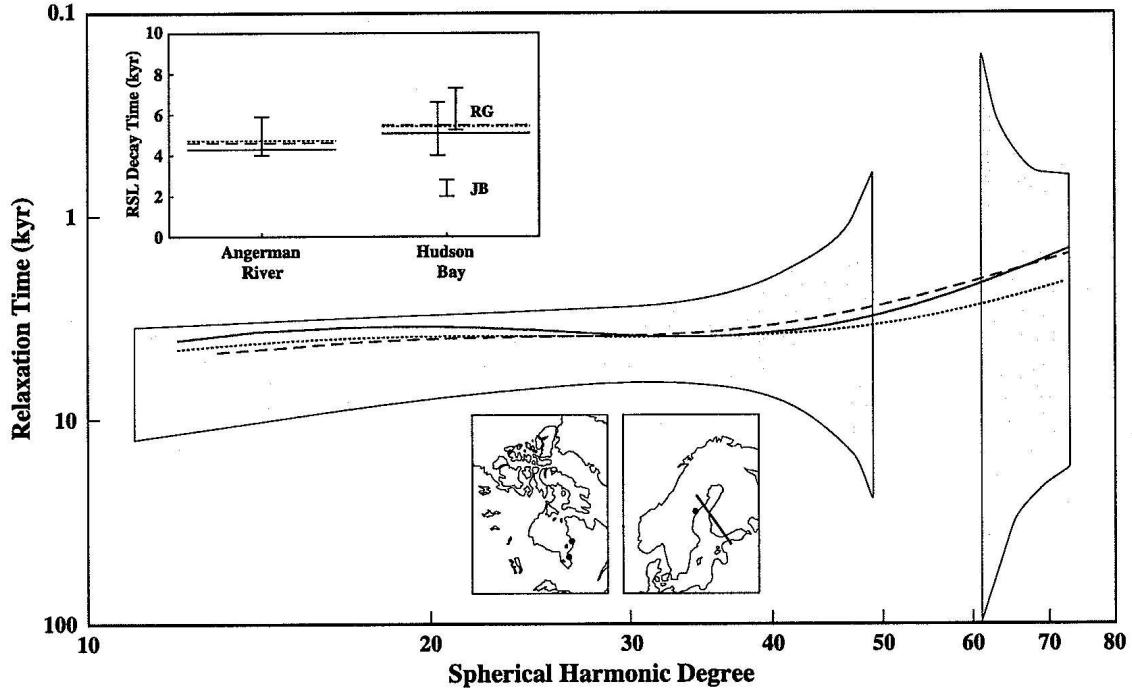


Fig. 2. A summary of two sets of postglacial rebound constraints on the viscosity inversions. Main frame—the relaxation spectrum derived [30] from the postglacial sea-level history in Fennoscandia along a profile shown in the bottom-right inset. Top-left inset—postglacial decay times at Angerman River, Sweden [42], as well as James Bay (JB) [44] and Richmond Gulf (RG) [43 (right datum), 44 (left datum)] in the Hudson Bay region. (Our inversions used the decay time derived in [44] for Richmond Gulf.) The location of these three sites is given in the insets at bottom. The dotted line superimposed on each datum represents a forward prediction based on the GIA-only inversion in Fig. 3, while the solid and dashed lines are predictions generated using the two joint GIA-convection viscosity inferences shown (using the same line type) in Fig. 3. For a given viscosity model, the decay times predicted at Richmond Gulf and James Bay are similar and we show only the average prediction for the two sites. The GIA forward predictions adopt an elastic lithospheric thickness, LT (see text), of 80 km.

Mitrovica J. X. & Forte A. M. (2004): A new inference of mantle viscosity based upon joint inversion of convection and glacial isostatic adjustment data, *EPSL.*, 225, 177–189.

Modèle de viscosité

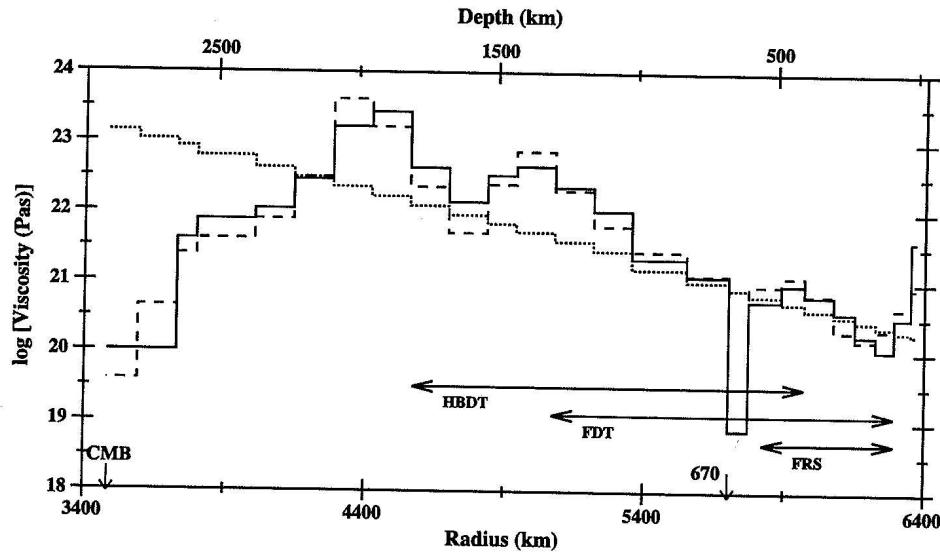


Fig. 3. Results of Occam-style inversions (see text for details) of the GIA and convection data sets described in the text. The dotted curve is the inference based on the inversion of GIA data alone. The solid and dashed lines are joint inversions of GIA and convection data sets distinguished on the basis of the seismic shear wave heterogeneity model adopted in the prediction of convection observables (Grand [54] and Ekström and Dziewonski [55], respectively). The solid and dashed profiles have a viscosity value of 7.8×10^{18} Pa s in the low-viscosity layer above 670 km depth. The three horizontal solid lines represent three of the radial regions resolved by the GIA data, as estimated in previous studies (see text). These include constraints imposed by: the Fennoscandian decay time (“FDT”), Hudson Bay decay times (“HBDT”) and the Fennoscandian relaxation spectrum (“FRS”).

(Mitrovica & Forte 2004)

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5. **Mitrovica J. X. & Forte A. M.** (2004):
A new inference of mantle viscosity based upon joint inversion of convection and glacial isostatic adjustment data, *EPSL.*, **225**, 177–189.
6. **Peltier W. R.** (1974): The impulse response of a Maxwell Earth, *Rev. Geoph. and Space Ph.*, **12**, 649–669.
7. **Yuen D. A. & Peltier W. R.** (1982): Normal modes of the viscoelastic Earth, *Geophys. J. R. astr. Soc.*, **49**, 495–526.
8. **Zürn W.** (1997): Earth tide observations and interpretation, in: *Lecture Notes in Earth Sciences - Tidal phenomena*, eds Wilhelm, Zürn, Wenzel, Springer.