TUTORIAL # 4

“ORBITES DE SATELLITES GÉODÉSIQUES DANS LE CHAMP DE GRAVITÉ TERRESTRE“

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Résumé

Ce cours a pour but, avec un minimum d’outils mathématiques simples, de familiariser le lecteur avec les équations fondamentales de la mécanique céleste (Gauss, Lagrange, Hill) et leur application au calcul analytique approché des perturbations subies par les trajectoires de satellites artificiels de la Terre, principalement dans le domaine de la géodésie spatiale.

“GEODETIC SATELLITE ORBITS IN THE EARTH’S GRAVITY FIELD“

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5• REFERENCES

Our goal is to give the minimum of what is necessary to understand the gross evolution of geodetic (i.e. low to medium altitude) satellite orbits around the Earth, as they are perturbed by the spatial variations of the gravitational field. Then an overview is given on how global geopotential models are determined. Finally an application is made on the derivation of orbital radial perturbations induced by the geopotential, which are fundamental for the planning and analysis of altimetry missions.

This does not pretend to be a course in celestial mechanics, and we will use the most simple mathematical tools whenever possible, still preserving the rigour of the proofs.

1• INTRODUCTION AND BASIC CONCEPTS

The motion of an Earth artificial satellite is the motion of a body with very small mass and negligible dimensions with respect to the planet. For most forces acting on the satellite, and especially the gravitational forces, it is sufficient to approximate it by a point mass. Surface forces, which are more complex due to their nature, require special treatments in which the shapes and surface properties of all the spacecraft elements are modelled ; they will not be treated in detail in this tutorial.

Other forces, such as the attraction of the Sun and Moon, the solid and fluid tidal effects will also be considered as being very small with respect to the main gravitational ones. Also, the equations of motion will be written, with sufficient approximations, in a reference system assumed to be fixed in space, corrections to this hypothesis being suitable of a treatment in terms of small perturbations. Finally, the actual law of forces will be written according to the classical mechanics ; the framework of general relativity is the rigourous one but differences with the classical approach are negligible for our purpose here.

Therefore, we consider that a massive point *O*, with mass *M*, exerts on a point mass *S*, of mass *m*, a force $\overline{F}_{o\to s} = -GmM\,\overline{OS}/OS^3$, (where *G* is the constant of gravitation and the overbar denotes a vector), and that the acceleration acquired by *S* is proportional to the force. This principle must however be written in a galilean reference system ($\mathcal{R}$). If we take a system of axis (∑) centered at *O*, with fixed directions in space, we write :

$$m\left[\frac{d^2\,\overline{OS}}{dt^2} + \overline{\Gamma}_{O/(\mathrm{R})}\right] = -GmM\frac{\overline{OS}}{OS^3}$$

and

$$M\overline{\Gamma}_{O/(\mathrm{R})} = GMm\,\overline{OS}/OS^3$$

Therefore :

$$\frac{d^2\,\overline{OS}}{dt^2} = -G\left(M+m\right)\frac{\overline{OS}}{OS^3} \tag{1}$$

Of course, in the case of an artificial satellite $m << M$ and $G\left(M+m\right)$ is replaced by *GM*.

1•1. THE UNPERTURBED SATELLITE ORBIT (two-body problem)

This is the orbit of *S* around a perfectly spherical Earth, of center of mass *O*. This point is one focus of the conic on which *S* moves ; in our case it will always be an ellipse (first Kepler's law). The closest point to *O* is the perigee *P*, the farthest is the apogee *A*, $\overline{OS}$ is the radius vector $\overline{r}$ of modulus *r*, the ellipse semi-major axis is *a*, its eccentricity is *e*, $p = a\left(1-e\ \right)$ is the parameter. *S* is positioned : either by the true anomaly $v = \left(\overline{OP},\overline{OS}\right)$, or by the eccentric anomaly $E = \left(\overline{OP},\overline{OS'}\right)$, where S' is on the principle circle (of which the ellipse is the affine transformed), or by the mean anomaly $M = \left(\overline{OP},\overline{OS''}\right)$ where S'' moves on the principal circle with a uniform velocity at the motion period, *T* (fig. 1).

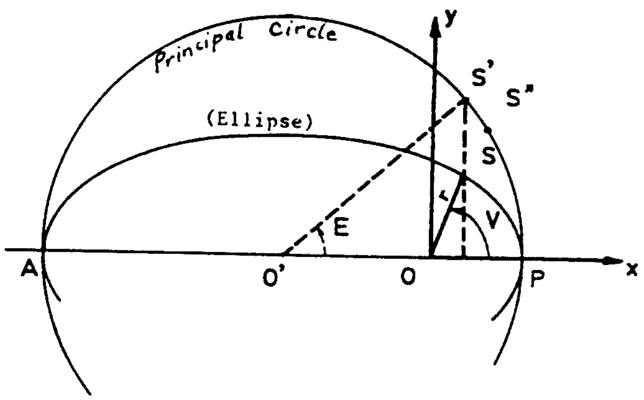

Fig. 1. The elliptical motion of *S* around *O*

In the plane of the orbit, we have for the radius *r* and coordinates *x*, *y* of *S* :

$$\begin{aligned} x &= r\cos v = a(\cos E - e) \\ y &= r\sin v = a\sqrt{1-e^2}\,\sin E \\ r &= \frac{a(1-e^2)}{1+e\cos v} = a(1-e\cos E) \end{aligned} \tag{2}$$

(from which sin *v*, cos *v* can be expressed in terms of sin *E*, cos *E*, and vice versa). From these, other useful formulas are derived :

$$\begin{aligned} tg\frac{v}{2} &= \sqrt{\frac{1+e}{1-e}}\,tg\frac{E}{2} \\ tg\frac{v-E}{2} &= \frac{\beta\sin E}{1-\beta\cos E} = \frac{\beta\sin v}{1+\cos v} \\ \beta &= e\Big/\left(1+\sqrt{1-e^2}\right) \\ \sin v - \sin E &= \beta\sin(v+E) \end{aligned} \tag{3}$$

The main motion *n* is defined by $n = 2\pi / T$ and we have :

$$M = n\left(t - t_o\right) \tag{4}$$

t_o being an epoch when S passes through perigee.

Finally we have the second Kepler's law :

$$r^2 \frac{dv}{dt} = na^2\sqrt{1-e^2} \qquad ,$$

the kinetic energy integral from which the velocity V is such that :

$$V^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

and Kepler's third law :

$$n^2 a^3 = GM = \mu$$

Practically, in order to compute x,y from t, it is necessary to compute M, then E from the Kepler equation :

$$E - e\sin E = M$$

then to apply (2).

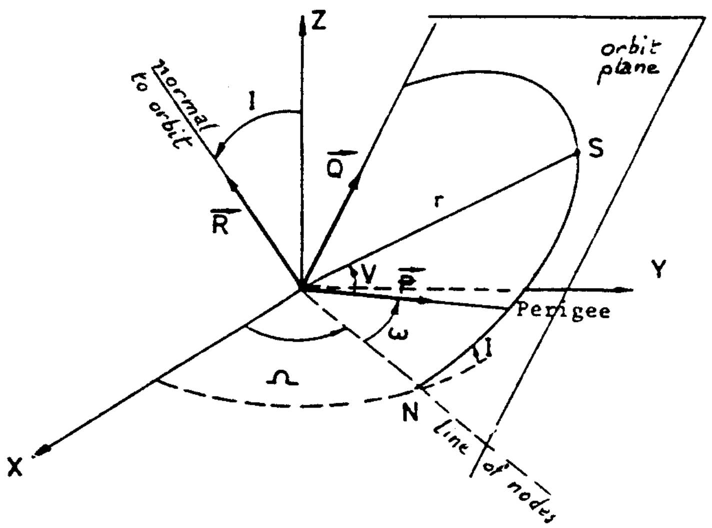

Fig. 2. The orbit in space

Actually, the orbit lies in a plane which position in space depends on the initial conditions. The plane intersection with the (X,Y) plane of (Σ) is the line of nodes, the ascending node N being the point where S crosses the equatorial plane with Z increasing ; the longitude of N is $\Omega = \left(\overline{OX}, \overline{ON}\right)$. The angle between the equatorial plane and the orbital plane is the inclination I, counted from 0° to 90° for direct motion and from 90° to 180° for retrograde motion. In the orbital plane, the direction of the perigee P is counted from N : $\omega = \left(\overline{ON}, \overline{OP}\right)$ is the argument of perigee. These angles are shown on figure 2.

Let us call $\mathcal{T}$ the transformation : $(a, e, I, \Omega, \omega, M) \rightarrow (\bar{r}, \dot{\bar{r}})$.

With :

$$\bar{r} = \left[X_1\ X_2\ X_3\right]^{+} = [XYZ]^{+}$$

we have :

$$\bar{r} = R_3(\Omega)\, R_1(I)\, R_3(\omega)\left[x\ y\ o\right] \tag{9}$$

where :

$$R_1(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix},\ R_2(\alpha) = \begin{pmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s & 0 & c \end{pmatrix},\ R_3(\alpha) = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$(c = \cos\alpha, s = \sin\alpha)$

x,y are of course given by (2).

The second part of the transformation is given by :

$$\dot{\bar{r}} = n\frac{a^2}{r}\left(-\sin E\,\overline{\mathbf{P}} + \cos E\sqrt{1-e^2}\ \overline{\mathbf{Q}}\right) \tag{10}$$

where :

$$\overline{\mathbf{P}} = \begin{pmatrix} \cos\Omega\cos\omega - \cos I\sin\omega\sin\Omega \\ \sin\Omega\cos\omega + \cos I\sin\omega\cos\Omega \\ \sin I\sin\omega \end{pmatrix}$$

and :

$$\overline{\mathbf{Q}} = \partial\mathbf{P} / \partial\omega$$

(to derive (10) from (9), one uses the fact that $dE / dt = n\,a / r$).

The inverse transformation T^{-1} can be achieved in different ways. One is the following :

$.a = \left(2/r - \dot{r}^2/\mu\right)^{-1}$

.compute $\overline{C} = \overline{r} \times \dot{\overline{r}}$ (angular momentum) ; $\left|\overline{C}\right| = C$ is constant

.I = $\cos^{-1}\left(C_3/C\right)$

$.k\sin\Omega = C_1,\;\; k\cos\Omega = -C_2$, with $k = \left(C_1^2 + C_2^2\right)^{1/2}$: define Ω.

.compute $p = C\sqrt{\mu a} \Rightarrow e = \left(1 - p/a\right)^{1/2}$

.compute $tg\gamma = \overline{r}.\dot{\overline{r}}\Big/\left[\mu a\left(1-e^2\right)\right]^{1/2}$, then E is defined by : $e\sin E = tg\gamma\left(1-e\right)^{1/2}$,

$e\cos E = 1 - r/a$, and $M = E - e\sin E$

.compute v from : $re\sin v = p\,tg\,\gamma,\;\; re\cos v = p - r$

then $(\omega + v)$ from : $r\sin I\sin(\omega + v) = X_3$

$r\sin I\cos(\omega + v) = \left(X_1\cos\Omega + X_2\sin\Omega\right)\sin I$

$\Rightarrow \omega.$

For future usage, we now need some series expansions of a few functions of the two-body problem. Such expansions may be viewed as power series in *e*, or as Fourier series in *M* depending on the utilization. By an application of a theorem by Lagrange applied to Kepler equation, it can be shown that any function *F* (*E*) of the eccentric anomaly *E* may be expanded as :

$$F(E) = F(M) + \sum_{n=1}^{\infty}\frac{e^n}{n!}\frac{d^{n-1}}{dM^{n-1}}\left[F'(M)\sin^n M\right] \tag{11}$$

which is convergent for $e < e_o, e_o$ being defined by $1 + \left(1+e_o^2\right)^{1/2} = e_o\exp\left(1+e_o^2\right)^{1/2}$, that is $e_o = 0.6627...$

We then find, for example :

$$E = M + \sum_{n=1}^{\infty}\frac{e^n}{2^{n-1}n!}\sum_{j=o}^{(n-1)/2}(-1)^j\binom{n}{j}\left(n-2j\right)^{n-1}\sin\left(n-2j\right)M \tag{12}$$

$$\frac{r}{a} = 1 - e\cos M - \frac{e^2}{2}(\cos 2M - 1) - \sum_{n=3}^{\infty}\frac{e^n}{2^{n-1}(n-1)!}\sum_{j=o}^{(n-1)/2}(-1)^j\binom{n}{j}\left(n-2j\right)^{n-2}\cos\left(n-2j\right)M \tag{13}$$

It is obvious that, since the motion is periodic, any function of coordinates is 2π-periodic in *M*. For example, cos *E* can be expanded as a cosine series in *M* :

$$\cos E = a_o + \sum_{p=1}^{\infty} a_p \cos pM$$

with :

$$a_o = \frac{1}{2\pi}\int_o^{2x} \cos E \, dM \,, \quad a_p = \frac{1}{\pi}\int_o^{2x} \cos E \cos p\, M \, dM$$

From $E - e\sin E = M$ we derive $dM = (1 - e\cos E)\,dE$; therefore : $a_o = -e/2$. For a_p we note that $\cos pM = \left[d(\sin pM)/dM\right]/p$ and we integrate by part, finding :

$$a_p = -\frac{1}{p\pi}\int_o^{2\pi} \sin pM \, d(\cos E)$$

Replacing *M* by *E* - *e* sin *E*, noting the invariance of the integral limits, we find :

$$a_p = \frac{1}{p}\frac{1}{2\pi}\int_o^{2\pi}\left[\cos(p-1)E - p\,e\sin E\right]dE - \frac{1}{p}\frac{1}{2\pi}\int_o^{2\pi}\left[\cos(p+1)E - p\,e\sin E\right]dE$$

We define the Bessel function of the first kind and of order *n* as :

$$J_n(x) = \frac{1}{2\pi}\int_o^{2\pi}\cos(nt - x\sin t)\,dt \tag{14}$$

So that :

$$a_p = \left[J_{p-1}(pe) - J_{p+1}(pe)\right]/p$$

and finally :

$$\cos E = -\frac{e}{2} + \sum_{p=1}^{\infty}\frac{1}{p}\left[J_{p-1}(pe) - J_{p+1}(pe)\right]\cos pM \tag{15}$$

A property of the Bessel functions is that :

$$J_{p-1}(x) - J_{p+1}(x) = 2J'_p(x) \tag{16}$$

which simplifies a little the form of (15).

From this it is easy to find that :

$$\frac{r}{a} = 1 + \frac{e^2}{2} - 2e\sum_{p=1}^{\infty}\frac{1}{p^2}\frac{d}{de}\left[J_p(pe)\right]\cos p\,M \tag{17}$$

Other useful formulas are :

$$v = M + 2\sum_{p=1}^{\infty}\frac{1}{p}\left\{J_p(pe) + \sum_{n=1}^{\infty}\beta^n\left[J_{p-n}(pe) + J_{p+n}(pe)\right]\right\}\sin pM \tag{18}$$

(β : as given in (3)).

$$\cos v = -e + 2\left(\frac{1-e^2}{e}\right)\sum_{p=1}^{\infty} J_p\left(pe\right)\cos p\, M \tag{19}$$

$$\sin v = 2\sqrt{1-e^2}\sum_{p=1}^{\infty}\frac{1}{p}\frac{d\left[J_p\left(pe\right)\right]}{de}\sin p\, M \tag{20}$$

More general formulas will be needed in the expression of the gravity field perturbations, for functions of the form $\left(r/a\right)^n \cos mv$ and $\left(r/a\right)^n \sin mv$ for any *n* and *m*, positive or negative. In complex form we write :

$$\left(\frac{r}{a}\right)^n \exp\left(imv\right) = \sum_{k=-\infty}^{+\infty} X_k^{nm} \exp\left(ikM\right) \tag{21}$$

This defines the Hansen coefficients. They are real functions of *e*, and can be evaluated in many different ways. Since X_k^{nm} is a Fourier coefficient, we have :

$$X_k^{nm} = \frac{1}{2\pi}\int_{-\pi}^{+\pi}\left(\frac{r}{a}\right)^n \exp\left(imv\right)\exp\left(-ikM\right)dM$$

Introducing $z = \exp(iE)$, we find that :

$$\exp(iM) = z\ \exp\left[-e\left(z - 1/z\right)/2\right],$$

$$dM = -i\left[1-\left(e/2\right)\left(z+1/z\right)\right]dz/z,$$

$$r/a = \left(1-\beta/z\right)\ \left(1-\beta z\right)/\left(1+\beta^2\right),$$

and $\exp(iv) = z\left(1-\beta/z\right)/\left(1-\beta z\right)$

Finally :

$$X_k^{nm} = \frac{1}{\left(1+\beta^2\right)^{n+1}}\frac{1}{2i\pi}\oint z^{m-k-1}\left(1-\frac{\beta}{z}\right)^{n+m+1}\left(1-\beta z\right)^{n-m+1}\cdot\exp\left[\frac{ke}{2}\left(z-\frac{1}{z}\right)\right]dz$$

From this, the following series expansion can be found (after some laborious algebra !) : for $k = m + s,$ with $s \geq o$

$$X_{m+s}^{n,m}(e) = (-1)^s \left(\frac{e}{2}\right)^s \sum_{t=o}^{\infty} \left\{ \sum_{j=o}^{t} \sum_{p=o}^{j} \binom{n+m+1}{j-p} \frac{(m+s)^p}{p!} \sum_{q=o}^{s+j} \binom{n-m+1}{s+j-q} \frac{(m+s)^q}{q!} (-1)^q \right.$$
$$\left. \left[2\binom{2t-n+s-p-q-2}{t-j} - \binom{2t-n+s-p-q-1}{t-j} \right] \right\} \left(\frac{e}{2}\right)^{2t} \quad (22)$$

If $s < o$, we compute $X_{m+s}^{nm} = X_{-m,-s}^{n,-m}$ by the same formula, using the property of symmetry of the Hansen functions $\left(X_k^{nm} = X_{-k}^{n,-m}\right)$.

In the above formula, binomial coefficients $\binom{-\mu}{p}$, where $\mu \geq o$, must be computed as being equal to $(-1)^p \binom{\mu+p-1}{p}$, p being always positive.

Another expression for the Hansen coefficients is :

$$X_k^{nm}(e) = \frac{1}{\left(1+\beta^2\right)^{n+1}} \sum_{p=o}^{\infty} \sum_{q=o}^{\infty} \binom{n+m+1}{p} \binom{n-m+1}{q} (-\beta)^{p+q} J_{k-m+p-q}(ke) \quad (23)$$

which can be more economical to evaluate than (22). From (22) and from the symmetry property, it is obvious that $X_k^{nm}(e) = o\left(e^{|k-m|}\right)$.

For large values of the *n*, *m*, *k* indices, it is numerically more efficient and precise to compute the Hansen coefficients by Fourier transform, from their definition (formula 21).

1•2. DISTURBING FORCES ON AN ARTIFICIAL SATELLITE

These forces are of different types :

- gravitational forces : first, and most important for our purpose here, are the forces due to the non-sphericity of the Earth in the general sense (geometrical form, internal density distribution). The main term is related to the flattening of our planet, the others describe all lateral density variations. This will be the whole subject of chapter 2. We then have :
    - ◊ the third-body perturbations due to the Moon, the Sun, and the closest and/or biggest planets,

- ◊ the tidal forces of various origins : solid tides due to the global yielding of the elastic Earth to the disturbing forces of the Sun and Moon ; ocean tides with numerous frequencies and varied amplitudes.
- ◊ atmospheric masses variations associated with pressure changes, and effects of these changes on the solid crust (by elastic deformations due to loading) and also on the ocean and large sea and lake surfaces.

- surface forces :
  - ◊ atmospheric drag, which acts in a very complex way due to the variations of the atmosphere density under the action of the Sun (solar cycle, yearly, seasonal, monthly, daily and hourly variations do exist due to the Sun activity, geomagnetic effects and induced chemical reactions), also due to the complex shapes of satellites and the nature of their surface elements,
  - ◊ radiation pressures : there is the direct solar pressure but also the one coming from the re-radiation of the Sun light by the Earth (albedo effect, the most complex since it is related to the cloud coverage), plus the infrared radiation of the Earth (considered as a black body), all requiring a careful modeling of the spacecraft components.

Finally, correction terms to the total acceleration of the satellite must be added to account for the correct relativistic description of the equations of motion and, if the reference frame in which these equations are written is moving, apparent accelerations have to be included.

1•3. EQUATIONS OF PERTURBED MOTION (LAGRANGE, GAUSS, HILL)

We have already written the cartesian equations of motion, (1), in the reference frame (Σ) in the case of the two body problem. With initial conditions $\left(\bar{r}_o, \dot{\bar{r}}_o\right)$ at t_o, this is a system of ordinary differential equations which solution is uniquely defined. Actually, (1) is equivalent to the system :

$$\frac{d\bar{\alpha}}{dt} = A\left(\bar{\alpha}, t\right)$$

with :

$\bar{\alpha} = [a, e, I, \Omega, \omega, M]$

$A = [o, o, o, o, o, n_o]$, $n_o = (\mu/a_o^3)^{1/2}$

a_o being computed from $(\bar{r}_o, \dot{\bar{r}}_o)$

This simply reflects the fact that, if one transforms the system (1) by (T) - given by (9) and (10), one finds the system for $\bar{\alpha}$. It is therefore quite natural, when introducing disturbing accelerations which are very small with respect to μ/r^2, to use the same transformation, hoping that the solution of the transformed system will be expressed as small variations around the solution of the two body problem, that is around

$$\bar{\alpha}_o = [a_o, e_o, I_o, \Omega_o, \omega_o, M_o + n_o(t - t_o)].$$

Let us write the equations of motion including the disturbing accelerations $\bar{\gamma}$ (one will often say disturbing "forces") as :

$$\ddot{\bar{r}} = -\mu\bar{r}/r^3 + \bar{\gamma} \tag{24}$$

with $\bar{r}(t_o) = \bar{r}_o$, $\dot{\bar{r}}(t_o) = \dot{\bar{r}}_o$.

We assume that the right-hand side member satisfies conditions such that (24) has always one and only one solution $[\bar{r}(t), \dot{\bar{r}}(t)]$ for $|t - t_o|$ large enough for our application (Cauchy-Arzela conditions).

At any time t in this interval, we can therefore apply the transformation $\mathcal{T}^{-1}$ to the solution of (24) and we get quantities $a(t), e(t), I(t), \Omega(t), \omega(t), M(t)$ which are no longer constant (or linear in time for M). These are called osculating elements. Their physical meaning is simple ; if, for $t' > t$ we suppress $\bar{\gamma}$, then the satellite motion obeys the system :

$t' > t : \ddot{\bar{r}} = -\mu\bar{r}/r^3$, with $\bar{r}(t) = \bar{r}$, $\dot{\bar{r}}(t) = \dot{\bar{r}}$,

of which the solution is : $a(t') = a(t)$, $e(t') = e(t)$, $I(t') = I(t)$, $\Omega(t') = \Omega(t)$, $\omega(t') = \omega(t)$, $M(t') = M(t) + [\mu/a(t)^3]^{1/2}(t - t')$, that is a keplerian ellipse. This ellipse passes through the point $S(t)$ of radius vector $\bar{r}$ and a mobile on it has the same velocity vector $\dot{\bar{r}}$, but the acceleration is different by construction (that is the term "osculating" is improper from the geometrical viewpoint).

Thus, using the variables $a(t), e(t), \ldots M(t)$ allows to visualize the trajectory evolution (e.g. rotation of the plane, of the line of apside : apogee-perigee, ...). Now, we want to deduce from (24) and from the formulas for T and T^{-1} the system verified by the osculating elements, which must be of the form :

$$\frac{d\bar{\alpha}}{dt} = \text{function of } \bar{\alpha} \text{ and } \bar{\gamma}$$

with :

$$\bar{\alpha}(t_o) = \bar{\alpha}_o = \mathrm{T}^{-1}(\bar{r}_o, \dot{\bar{r}}_o)$$

The perturbing acceleration is projected on the mobile reference system axis : $(\hat{r}, \hat{s}, \hat{w})$ defined by : $\hat{r} = \bar{r}/r, \hat{s}$: unit vector orthogonal to $\hat{r}$ in the osculating plane and in the direction of $\dot{\bar{r}}, \hat{w} = \hat{r} \times \hat{s}$; that is $\hat{w} = (\bar{r} \times \dot{\bar{r}})/\|\bar{r} \times \dot{\bar{r}}\|$, and $\hat{s} = \hat{w} \times \hat{r}$ (fig. 3).

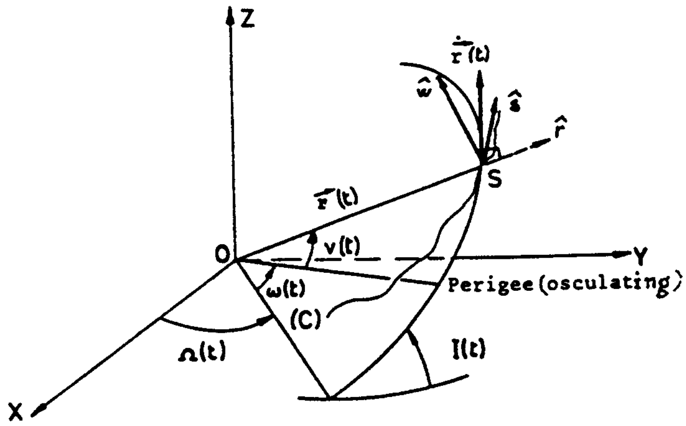

Fig. 3. The Gauss mobile system $(\hat{r}, \hat{s}, \hat{w})$

So : $\bar{\gamma} = \mathbf{R}\hat{r} + \mathbf{S}\hat{s} + \mathbf{W}\hat{w}$. Now, by derivation of (6) with respect to time and since $r\dot{r} = \bar{r} \cdot \dot{\bar{r}}$ (from $r^2 = \bar{r} \cdot \bar{r}$), we readily find :

$$(\mu/a^2)\dot{a} = 2\dot{\bar{r}} \cdot \bar{\gamma}$$

Rewriting (10) as $\dot{\bar{r}} = na\left(1-e^2\right)^{-1/2}\left[e\sin v\hat{r} + (1+e\cos v)\hat{s}\right]$, obtained by a rotation around $\hat{w}$ with angle v, we obtain :

$$\dot{a} = (2/n)\left(1-e^2\right)^{-1/2}\left[e\sin v\mathbf{R} + (1+e\cos v)\mathbf{S}\right]$$

We then use the angular momentum vector $\bar{C} = C\hat{w}$, with $C = \left[\mu a\left(1-e^2\right)\right]^{1/2} = (\mu p)^{1/2}$; $\bar{C}$ verifies $d\bar{C}/dt = \bar{r}\times\left(-\mu\bar{r}/r^3 + \bar{\gamma}\right) = \bar{r}\times\bar{\gamma} = 1/2\left(\mu/p\right)^{1/2}\dot{p}\hat{w} + (\mu p)^{1/2}\dot{\hat{w}}.$

We define the following unit vectors : $\bar{N}$ in the ascending node direction, $\bar{N}'$ orthogonal to $\bar{N}$ in the equatorial plane, $\bar{M}$: orthogonal to $\bar{N}$ in the (osculating) orbit plane, and $\bar{i}, \bar{j}, \bar{k}$: unit vectors of the (Σ) frame (fig. 4).

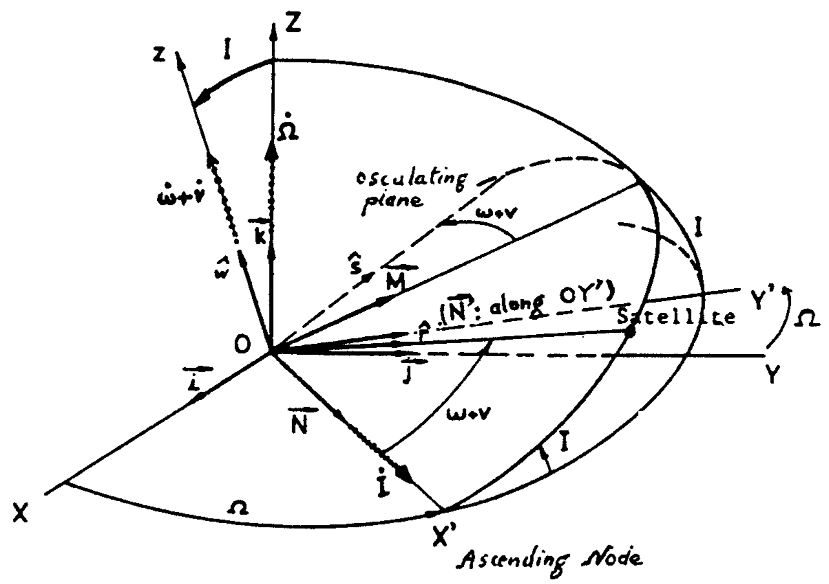

Fig. 4. The intermediate vectors $\bar{N}, \bar{N}', \bar{M}$ introduced for the equations for e, I, Ω

Writing $\bar{r}\times\bar{\gamma} = -r\mathbf{W}\hat{s} + r\mathbf{S}\hat{w}$, $\hat{w} = \bar{N}\times\bar{M}, \bar{M} = \bar{N}'\cos I + \bar{k}\sin I$, noting that $\bar{N}$ depends only on Ω, that $\dot{\bar{N}}' = \dot{\Omega}\, d\,\bar{N}'/d\Omega = -\bar{N}\dot{\Omega}$, also that $\dot{\bar{N}} = \bar{N}'\dot{\Omega}$, and taking account that $\bar{N}'\times\bar{k} = \bar{N}, \bar{N}\times\bar{N}' = \bar{k}, \bar{N}\times\bar{k} = -\bar{N}'$, we obtain :

$\dot{\hat{w}} = \dot{\Omega} \sin I \, \overline{N} - \dot{I} \, \overline{M}$

Noting also that $\hat{r} = \overline{N} \cos(\omega + v) + \overline{M} \sin(\omega + v)$, $\hat{s} = -\overline{N} \sin(\omega + v) + \overline{M} \cos(\omega + v)$, and equating the components of $d\overline{C}/dt$ on $\hat{r}, \hat{s}, \hat{w}$, we find three equations for $\dot{p} = \dot{a}(1 - e^2) - 2ae\dot{e}$, $\dot{I}$ and $\dot{\Omega}$.

The equation for $\dot{M}$ is obtained through : $r = r(a, e, M)$ which implies :

$\dot{r} = \dot{a} \, \partial r/\partial a + \dot{e} \, \partial r/\partial e + \dot{M} \, \partial r/\partial M$

It is easy to find $\dot{r} = nae \sin v/(1 - e^2)^{1/2}$, $\partial r/\partial a = r/a$, $\partial r/\partial E = ae \sin E$, $\partial E/\partial e = a \sin E/r$, $\partial E/\partial M = a/r$, $\partial r/\partial e = -a \cos v$, $\partial r/\partial M = ae \sin v/(1 - e^2)^{1/2}$, and this yields the equation for $\dot{M}$.

The last equation, for $\dot{\omega}$, is more tricky. We start from $\psi = \omega + v$, and from the second Kepler's law : $\dot{\psi} = na^2 (1 - e^2)^{1/2} / r^2$ which is valid in the osculating motion if ψ is counted from a fixed direction. But, in the real motion, all elements vary and the direction *ON* from which one would like to count ψ varies too ! Therefore, we cannot write $\dot{\psi} = \dot{\omega} + \dot{v}$ if we apply Kepler's law. We derive $\dot{\psi}$ directly from $tg\psi = \eta/\xi$, that is $d\psi = (\xi d\eta - \eta d\xi)/(\xi^2 + \eta^2)$, where $\overline{OS} = \xi \overline{N} + \eta \overline{M}$. For getting $d\xi, d\eta$, we compute $d\overline{OS}/dt$ when $\Omega, \omega + v$ and I vary ; in this case :

$d\overline{OS}/dt = [\dot{\Omega} \overline{k} + \dot{I} \overline{N} + (\dot{\omega} + \dot{v}) \hat{w}] \times \overline{OS}$

Writing this equality in $(\overline{N}, \overline{M}, \hat{w})$ with $\overline{k} = \sin I \, \overline{M} + \cos I \, \hat{w}$, we arrive at :

$\dot{\xi} = -r \sin(\omega + v)(\dot{\omega} + \dot{v} + \dot{\Omega} \cos I)$

$\dot{\eta} = r \cos(\omega + v)(\dot{\omega} + \dot{v} + \dot{\Omega} \cos I)$

(the 3rd equation would give $\dot{\zeta} ==$ (component of $d\overline{OS}/dt$ on $\hat{w}$) $= \eta \dot{I} - \xi \sin I \dot{\Omega}$) ; we then find :

$d\psi = d\omega + dv + d\Omega \cos I$

Consequently :

$\dot{\omega} = na^2 (1 - e^2)^{1/2} / r^2 - \dot{v} - \dot{\Omega} \cos I$

$\dot{v}$ is computed as $dv(e,M)/dt = \dot{e}\,\partial v/\partial e + \dot{M}\,\partial v/\partial M$, with :
$\partial v/\partial e = \sin v\ \left[a/r + 1/(1-e^2)\right]$, $\partial v/\partial M = (1-e^2)^{1/2}\, a^2/r^2$. The equation for $\dot{\omega}$ follows ...

We now summarize the six equations, known as Gauss equations, which are obtained by the elementary manipulations shown above :

$$\begin{aligned}
\dot{a} &= 2\left[\mathbf{R} e\sin v + \mathbf{S}(1+e\cos v)\right]/(nf)\\
\dot{e} &= f\left[\mathbf{R}\sin v + \mathbf{S}(\cos E + \cos v)\right]/(na)\\
\dot{I} &= \mathbf{W} r\cos(\omega+v)/\left(na^2 f\sin I\right)\\
\dot{\Omega} &= \mathbf{W} r\sin(\omega+v)/\left(na^2 f\right)\\
\dot{\omega} &= f\Big[-\mathbf{R}\cos v + \mathbf{S}\left(1+(1+e\cos v)^{-1}\sin v\right.\\
&\qquad - \mathbf{W} r\cos I\sin(\omega+v)/\left(na^2 f\sin I\right)\Big]/(nae)\\
\dot{M} &= n + f^2\left\{\mathbf{R}\left[-2e/(1+e\cos v)+\cos v\right] - \mathbf{S}\left[1+(1+e\cos v)^{-1}\right]\sin v\right\}/(nae)
\end{aligned} \tag{25}$$

(here : $f = \sqrt{1-e^2}$).

Next, we will derive the Lagrange equations. They are a particular case of the Gauss equations when the disturbing acceleration $\bar{\gamma}$ is the gradient of a function $\mathcal{R}$ (force function) : $\bar{\gamma} = \overline{\nabla}\mathrm{R}$. This is the case of all forces of gravitational origin and this leads to a simpler differential system. In the $(\hat{r},\hat{s},\hat{w})$ system we can write :

$$d\mathrm{R} = \overline{\nabla}\mathrm{R}\cdot d\,\overline{OS} = \mathbf{R}dr + Sr\,d\psi + \mathbf{W}d\zeta$$

or, for any orbital element α :

$$\frac{\partial\mathrm{R}}{\partial\alpha} = \mathbf{R}\frac{\partial r}{\partial\alpha} + \mathbf{S}r\frac{\partial\psi}{\partial\alpha} + \mathbf{W}\frac{\partial\zeta}{\partial\alpha'}$$

where $(dr, rd\psi, d\zeta)$ are the orthogonal components of $d\,\overline{OS}$. Clearly, *dr* can be due to changes *da, de, dM* only and we have : $\partial r/\partial a = r/a$, $\partial r/\partial e = -a\cos v$, $\partial r/\partial M = a\,e\sin v/(1-e^2)^{1/2}$, already used above. Similarly, $d\psi$ can result from changes in Ω, ω, and v as shown in the derivation of the Gauss equation for ω, and we have : $\partial\psi/\partial\omega = 1$, $\partial\psi/\partial\Omega = \cos I$, $\partial\psi/\partial v = 1$;

and since v is function of e and M, $\partial\psi/\partial e = \sin v\left[a/r + 1\left(1-e^2\right)^{-1}\right]$, $\partial\psi/\partial M = \left(a/r\right)^2\left(1-e^2\right)^{1/2}$. Finally, we already obtained $d\zeta = \eta dI - \xi \sin I d\Omega$, from which $\partial\zeta/\partial I = \eta = r\sin(\omega+v)$ and $\partial\zeta/\partial\Omega = -\xi\sin I = -r\cos(\omega+v)\sin I$.

All the other partial derivatives are equal to zero. Therefore we have found :

$$\frac{\partial \mathrm{R}}{\partial a} = \frac{r}{a}\mathbf{R}$$

$$\frac{\partial \mathrm{R}}{\partial e} = -\mathbf{R}a\cos v + S\sin v\left(a + \frac{r}{1-e^2}\right)$$

$$\frac{\partial \mathrm{R}}{\partial I} = r\sin(\omega+v)\,\mathbf{W} \tag{26}$$

$$\frac{\partial \mathrm{R}}{\partial \Omega} = r\mathbf{S}\cos I - r\mathbf{W}\cos(\omega+v)\sin I$$

$$\frac{\partial \mathrm{R}}{\partial \omega} = r\mathbf{S}$$

$$\frac{\partial \mathrm{R}}{\partial M} = \mathbf{R}\frac{ae\sin v}{\sqrt{1-e^2}} + \mathbf{S}\frac{a^2}{r}\sqrt{1-e^2}$$

We now transform the Gauss equations one by one. For $\dot{a}$ we replace $1+e\cos v$ by $a\left(1-e^2\right)/r$ and relate the right-hand side to $\partial\mathrm{R}/\partial M$. For $\dot{e}$, we note that $\cos E + \cos v = \left[a\left(1-e^2\right)/r - r/a\right]/e$, we have $\mathbf{R}$ sin v in terms of $\partial\mathrm{R}/\partial M$ and $\mathbf{S}$, which we replace by $\left(1/r\right)\partial\mathrm{R}/\partial\omega$. $\dot{I}$ is obtained from $\partial\mathrm{R}/\partial\Omega$ and $\partial\mathrm{R}/\partial\omega$. $\dot{\Omega}$ is immediately written in terms of $\partial\mathrm{R}/\partial I$. The first two terms in the bracket for $\dot{\omega}$ equal $\left(1/a\right)\partial\mathrm{R}/\partial e$ and the last one is proportional to $\partial\mathrm{R}/\partial I$. For $\dot{M}$, we first express $\mathbf{S}$ and its factor in terms of $\mathbf{R}$ and $\partial\mathrm{R}/\partial e$ and then replace $\left(1-e^2\right)\mathbf{R}/(1+e\cos v)$ by $\left(r/a\right)\mathbf{R} = \partial\mathrm{R}/\partial a$.

Finally, the six Lagrange equations are :

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial \mathrm{R}}{\partial M}$$

$$\frac{de}{dt} = \frac{1-e^2}{na^2 e}\frac{\partial \mathrm{R}}{\partial M} - \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial \mathrm{R}}{\partial \omega}$$

$$
\begin{aligned}
\frac{dI}{dt} &= \frac{\cos I}{na^2\sqrt{1-e^2}\,\sin I}\frac{\partial R}{\partial \omega} - \frac{1}{na^2\sqrt{1-e^2}\,\sin I}\frac{\partial R}{\partial \Omega} \\
\frac{d\Omega}{dt} &= \frac{1}{na^2\sqrt{1-e^2}\,\sin I}\frac{\partial R}{\partial I} \\
\frac{d\omega}{dt} &= \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial e} - \frac{\cos I}{na^2\sqrt{1-e^2}\,\sin I}\frac{\partial R}{\partial I} \\
\frac{dM}{dt} &= n - \frac{2}{na}\frac{\partial R}{\partial a} - \frac{1-e^2}{na^2 e}\frac{\partial R}{\partial e}
\end{aligned}
\tag{27}
$$

The form of this system is remarquable. If we take $\sigma = M - nt$ instead of M, we have $\partial R/\partial M = \partial R/\partial \sigma$ and :

$$
\frac{d}{dt}\left[a, e, I, \Omega, \omega, \sigma^{+}\right] = M\left(a, e, I\right)\left[R'_a, R'_e, R'_I, R'_\Omega, R'_\omega, R'_M\right]^{+}
$$

where $R'_\alpha = \partial R/\partial \alpha$ and where n is replaced by $\left(\mu/a^3\right)^{1/2}$. $\mathcal{M}$ is an antisymmetric matrix with only ten non-zero elements. The system may be simplified further if one adopts the so-called Delaunay variables :

$$
\begin{aligned}
L &= \sqrt{\mu a} \\
G &= \sqrt{\mu a\left(1-e^2\right)} \\
H &= \sqrt{\mu a\left(1-e^2\right)}\cos I \\
l &= M \\
g &= \omega \\
h &= \Omega
\end{aligned}
\tag{28}
$$

In this case, we simply have, with $F = R + \mu^2/\left(2L^2\right)$

$$
\begin{aligned}
\frac{dL}{dt} = \frac{\partial F}{\partial l} \quad &, \quad \frac{dG}{dt} = \frac{\partial F}{\partial g} \quad , \quad \frac{dH}{dt} = \frac{\partial F}{\partial h} \\
\frac{dl}{dt} = -\frac{\partial F}{\partial L} \quad &, \quad \frac{dg}{dt} = -\frac{\partial F}{\partial G} \quad , \quad \frac{dh}{dt} = -\frac{\partial F}{\partial H}
\end{aligned}
\tag{29}
$$

This system is said to be canonical, with the hamiltonian $\mathcal{F}$. It is the best suited one for some sophisticated techniques of deriving analytical solutions.

In the case of quasi-circular orbits, it may be of interest to describe the real motion in terms of discrepancies with respect to a reference circular trajectory whose plane is fixed in (Σ) and defined by its mean motion $\tilde{n}$, the radius of the orbit, $\tilde{r}$, satisfying Kepler's 3rd. law : $\tilde{n}^2\tilde{r}^3 = \mu$.

The true position *S* of the satellite will be given by its three coordinates (u, v, w) in the mobile system rotating with the fictitious reference point $\tilde{S}$ (fig. 5).

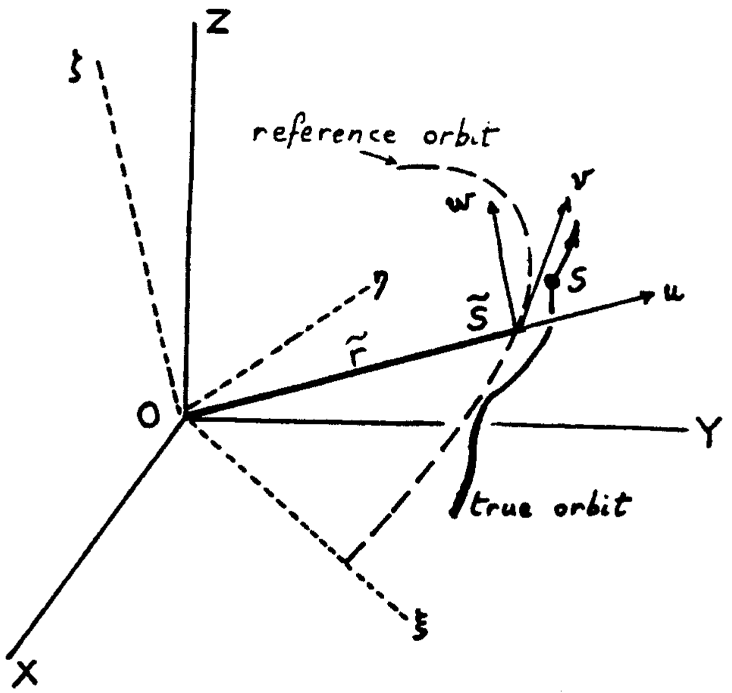

Fig. 5. The Hill reference orbit and rotating system

We here restrict ourselves to conservative forces, that is $\bar{\gamma} = \overline{\nabla}\mathrm{R}$. In the rotating system $\tilde{\Sigma} = \{uvw\}$, which rotation vector $\bar{\rho}$ with respect to (_) is $\tilde{n}\bar{w}$, we have :

$$- \mu\bar{r}/r^3 + \bar{\gamma} = \left[d^2\bar{r}/dt^2\right]_\Sigma = \left[d^2\bar{r}/dt^2\right]_{\tilde{\Sigma}} + 2\bar{\rho}\times\left[d\bar{r}/dt\right]_{\tilde{\Sigma}} + \bar{\rho}\times(\bar{\rho}\times\bar{r}) + \dot{\bar{\rho}}\times\bar{r}$$

The last term is equal to zero since $\tilde{n}$ and therefore $\bar{\rho}$ are constant. This equation is projected on $\tilde{\Sigma}$, in which the coordinates of *S* are $\tilde{r} + u, v, w$. We find :

$$\ddot{u} - 2\tilde{n}\dot{v} - \tilde{n}^2(\tilde{r} + u) = -\frac{\mu}{r^3}(\tilde{r} + u) + \frac{\partial R}{\partial u}$$

$$\ddot{v} + 2\tilde{n}\dot{u} - \tilde{n}^2 v = -\frac{\mu}{r^3} v + \frac{\partial R}{\partial v}$$

$$\ddot{w} = -\frac{\mu}{r^3} w + \frac{\partial R}{\partial w}$$

Hill equations are finally obtained by linearizing this system around $u = v = w = o$. We first write : $r^2 = (\tilde{r} + u)^2 + v^2 + w^2 \approx \tilde{r}^2 + 2u\tilde{r} = \tilde{r}^2\left(1 + 2u/\tilde{r}\right)$, from which : $r^{-3} \approx \tilde{r}^{-3}\left(1 - 3u/\tilde{r}\right)$. Hence, the first term in the right hand side members of the above equations becomes $-\mu(\tilde{r} - 2u)/\tilde{r}^3, -\mu w/\tilde{r}^3, -\mu w/\tilde{r}^3$. Replacing μ by $\tilde{n}^2\tilde{r}^3$ yields the final Hill system :

$$\begin{aligned} &\ddot{u} - 2\tilde{n}\dot{v} - 3\tilde{n}^2 u = \partial R/\partial u \\ &\ddot{v} + 2\tilde{n}\dot{u} = \partial R/\partial v \qquad (30) \\ &\ddot{w} + \tilde{n}^2 w = \partial R/\partial w \end{aligned}$$

Note that the last equation is decoupled from the others, allowing a separate treatment.

1•4. APPROXIMATE ANALYTICAL SOLUTIONS OF THE EQUATIONS OF MOTION

For further use in this course, it is sufficient to consider only the case of the Lagrange equations with a disturbing force fonction $\mathcal{R}$. However, much of what follows may be applied to other cases treated with the Gauss system.

$\mathcal{R}$ is a function of the position, hence of the six osculating elements, and also of the coordinates of the disturbing bodies (Moon, Sun). It is a 2π-periodic function in Ω, ω, M since it must have the same numerical value when these arguments change by 2π, the others being constant. On the other hand, the position of a disturbing body may be expressed via the orbital elements of its trajectory : $a^*, e^*, I^*, \Omega^*, \omega^*, M^*$, with respect to the reference frame (Σ) or to an intermediate reference frame with given (slowly varying) Euler angles $\varepsilon = \left(\varepsilon_1, \varepsilon_2, \varepsilon_3\right)$. It will be assumed, with enough accuracy here, that $\left(a^*, e^*, I^*\right)$ are constant and that Ω^*, ω^*, M^* and ε are linear functions of time. Obviously, $\mathcal{R}$ must also be 2π-periodic in Ω^*, ω^*, M^* and in these Euler angles. Finally, and as concerned all Earth gravity direct and tidal effects, $\mathcal{R}$ must be 2π-periodic in θ, the sidereal time (we here assume that the equatorial plane of (Σ) is the Earth mean equator and that θ is the mean sidereal time, discrepancies from this hypothesis being treated as very small corrections to the solution).

Therefore, $\mathcal{R}$ may be expanded as a Fourier series of the form :

$$\mathrm{R} = \sum B_{jkl\, j^* k^* l^* pq}\left(a, e, I, a^*, e^*, I^*\right)\cos\left(j\Omega + k\omega + lM + j^*\Omega^* + k^*\omega^* + l^* M^* + p\theta + q\varepsilon + \Phi\right)$$

The summation runs on all indices, for all disturbing bodies, and the phase Φ is a function of these indices in general. We write $\mathcal{R}$ in a more compact form, as :

$$\mathrm{R} = \sum B_{ii^*h}\left(m, m^*\right)\cos\left(iA + i^* A^* + hH\right) \tag{31}$$

where i stands for $\left(j, k, l\right)$, i^* for $\left(j^*, k^*, l^*\right)$, h for $\left(p, q\right)$. m is the triplet (a,e,I) of the satellite metric[1] elements, A the triplet $\left(\Omega, \omega, M\right)$ of its angular elements (Although I is an angle, it is called a "metric" element similar to a,e, because of the type of equation which governs its behaviour.); m^*, A^* designate the metric and angular elements of a disturbing body ; and H stands for all other angular parameters (and the phase is distributed among all pertinent arguments and indices).

The form of the Lagrange equations is such that :

$$\frac{dm}{dt} = \sum C_{ii^*h}\left(m, m^*\right)\sin\left(iA + i^* A^* + hH\right) \tag{32}$$

$$\frac{dA}{dt} = \sum D_{ii^*h}\left(m, m^*\right)\cos\left(iA + i^* A^* + hH\right) \tag{33}$$

In (33) there generally exist terms with all indices equal to zero, that is terms which are independent of the angular elements : $D_{ooo}\left(m, m^*\right)$.

Many different methods exist for obtaining the solution of these equations. Modern approaches all use algebric manipulators, but it is usely not too difficult to obtain by hand calculations a first good idea of the solution characteristics by retaining only the most significant terms, in particular by neglecting all the terms that are of the order of the square of small quantities characterizing the disturbing function. Such a procedure is called a first order solution and is simple to apply once the equations are written as in (32) and (33).

Let us note $m_o = (a_o, e_o, I_o)$ the mean values of (a,e,I), which are obtained if one neglects all the terms in (32). These are substituted in (33) in which we also provisionally neglect the periodic terms, keeping only the D_{ooo}'s. We find : $dA/dt = D_{ooo}$, or : $\dot{\Omega}^{(o)} = n_\Omega(m_o)$, $\dot{\omega}^{(o)} = n_\omega(m_o), \dot{M}^{(o)} = n_M(m_o)$. The superscript ($o$) indicates that this is the beginning of the process of successive approximations. Actually, n_M consists of $n_o = (\mu/a_o^3)^{1/2}$ and of the term coming from the development. Integrating these equations, we obtain :

$$\begin{aligned} &\overline{\Omega} = n_\Omega (t - t_o) + \Omega_o \\ &\overline{\omega} = n_\omega (t - t_o) + \omega_o \quad \Leftrightarrow \quad \overline{A} = n_A (t - t_o) + A_o \\ &\overline{M} = n_M (t - t_o) + M_o \end{aligned} \tag{34}$$

These are linear, hence unbounded functions of time ; they are called secular terms and are the largest perturbations.

The next step of the process is to substitute m_o and $\overline{A}$ in the right hand sides of (32), (33), taking also into account that $m^*(t) \approx m_o^*,\ A^*(t) \approx \vec{A}^* = n_{A^*}(t - t_o) + A_o^*$, $H(t) \approx \overline{H} = n_H(t - t_o) + H_o$.

After integration, we obtain :

$$m = m_o - \sum \frac{C_{ii^*k}}{in_A + i^* n_{A^*} + hn_H} \cos\left(i\overline{A} + i^*\overline{A}^* + h\overline{H}\right) \tag{35}$$

$$A = \overline{A} + \sum \frac{D_{ii^*k}}{in_A + i^* n_{A^*} + hn_H} \sin\left(i\overline{A} + i^*\overline{A}^* + h\overline{H}\right) \tag{36}$$

Of course, the coefficients C_{ii^*k}, D_{ii^*k} are different for each of the metric or angular elements.

In this procedure, we have overlooked the fact that the first term of dM/dt is n and not n_o. We have $n^2a^3 = \mu$, from which $2\Delta n/n + 3\Delta a/a = o$, hence to first order : $n = n_o\left[1 - 3/2\left(\Delta a/a\right)\right]$. From (35) we get :

$$\Delta a = -\sum \frac{C^{(a)}_{ii^*k}}{in_A + i^* n_{A^*} + hn_H} \cos\left(i\overline{A} + i^*\overline{A}^* + h\overline{H}\right)$$

Then :

$$n = n_o + \frac{3n_o}{2a_o} \sum \frac{C^{(a)}_{ii^*k}}{in_A + i^* n_{A^*} + hn_H} \cos\left(i\overline{A} + i^*\overline{A}^* + h\overline{H}\right)$$

So we must add to the solution in M given by (36), with $D^{(M)}_{ii^*k}$, the integral of these additional terms, that is :

$$\Delta' M = \frac{3n_o}{2a_o} \sum \frac{C^{(a)}_{ii^*k}}{\left(in_A + i^* n_{A^*} + hn_H\right)^2} \sin\left(i\overline{A} + i^*\overline{A}^* + h\overline{H}\right) \qquad (37)$$

All periodic terms in (35), (36), (37) look similar. They are usually grouped in short, middle, long period terms depending on their period $2\pi/\left(in_A + i^* n_{A^*} + hn_H\right)$ with respect to the mean period of the satellite $2\pi/n_o$. It is interesting to note that there may exist combinations of the indices i, i^*, h such that, for n_A, n_{A^*}, n_H being given, the divisor $in_A + i^* n_{A^*} + hn_H$ becomes very small with respect to C_{ii^*k} or D_{ii^*k}, thus enhancing greatly the perturbation. This is called a resonance phenomenon. When the divisor becomes too small, the linear theory outlined above becomes meaningless and other techniques are required.

If one stops the procedure at the stage of the last equations, we usually do not have a full 1st. order theory with respect to the small parameter (s) of $\mathcal{R}$. There exist additional first order terms coming from the next step, that is when one substitutes m and A as given by (35), (36) and also (37), in the Lagrange equations and integrate again.

Finally, the form of the Lagrange (or Gauss) equations is such that orbits with small eccentricity and/or with small sin I cannot be properly treated without care. Either one must adopt another set of variables, such as $\left(e\sin\omega, e\cos\omega\right)$ instead of $\left(e, \omega\right)$ - for which there exist an equivalent system of equations, or one must expand the solution in the vicinity of $e = o$, or of $\sin I = o$, and properly re-arrange or group some terms (... which we will do later when dealing with quasi-circular orbits).

2• THE GEOPOTENTIAL AND ITS REPRESENTATION

2•1. SPHERICAL HARMONIC REPRESENTATION OF THE GEOPOTENTIAL

Let us consider the Earth (*E*) with its actual shape (grossly approximated by an ellipsoid of revolution) and its density distribution such that, at the current point P', the mass element is $dM' = \rho(P')\,dV'$ in the elementary volume dV'. Let (Σ_o) be a reference system fixed in (*E*). The gravitational force at any point *S* outside (*E*) derives from the force function (called geopotential) :

$$U = G\int_{(E)} dM'/\Delta \qquad (38)$$

where Δ is the distance SP' (fig. 6).

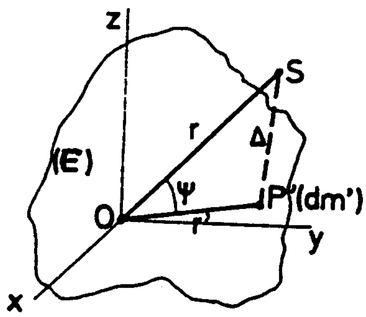

Fig. 6. The Earth and satellite point *S*

$1/\Delta$ is written as $r^{-1}\left[1-2(r'/r)\cos\psi+(r'/r)^2\right]^{-1/2}$. Now, if $r' < r$ for all P', the term $\left[1-2t\cos\psi+t^2\right]^{-1/2}$ with $r'/r = t < 1$, can be expanded in a convergent Legendre series :

$$\left[1-2t\cos\psi+t^2\right]^{-1/2} = \sum_{l=o}^{\infty} t^l P_l(\cos\psi) \qquad (39)$$

where $P_l(x) = \left\{ d^l \left[(x^2 - 1)^l \right] / dx^l \right\} / (2^l l!)$ is the usual Legendre polynomial of degree *l*. Then $P_l(\cos\psi)$ can be transformed as follows. Denoting by (ϕ', λ') the latitude and longitude of *P'* and by (ϕ, λ) those of *S*, we have :

$$\cos\psi = \sin\phi \sin\phi' + \cos\phi \cos\phi' \cos(\lambda - \lambda')$$

which is transformed by the operator P_l as (Legendre addition formula) :

$$P_l(\cos\psi) = \sum_{m=-l}^{+l} (-1)^m P_{lm}(\sin\phi) P_{l,-m}(\sin\phi') \exp[im(\lambda - \lambda')] \tag{40}$$

In this formula, $P_{lm}(x)$ is the Legendre associated function of the first kind, of degree *l* and order *m*, and is defined by :

$$m = o : P_{lo}(x) = P_l(x)$$

$$m > o : P_{lm}(x) = (1 - x^2)^{m/2} d^m [P_l(x)] / dx^m$$

$$P_{l,-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_{lm}(x)$$

M being the mass of the Earth (and since $P_o(x) = 1$), we obtain :

$$U = \frac{GM}{r} + \frac{G}{r} \sum_{l=1}^{\infty} \left(\frac{1}{r}\right)^l \sum_{m=-l}^{+l} \left[(-1)^m \int_{(E)} r'^l P_{l,m}(\sin\phi') \exp(-im\lambda') dM' \right] P_{lm}(\sin\phi) \exp(im\lambda)$$

This expansion requires that *S* be exterior to the smallest sphere containing (*E*), let us say a sphere of radius *R*. Introducing it as a factor of homogeneity, we obtain :

$$U = \frac{GM}{r} + \mathsf{R}$$

$$\mathsf{R} = \frac{GM}{r} \sum_{l=1}^{\infty} \left(\frac{R}{r}\right)^l \sum_{m=-l}^{+l} K_{lm} P_{lm}(\sin\phi) \exp(im\lambda) \tag{41}$$

with :

$$K_{lm} = \frac{(-1)^m}{MR^l} \int_{(E)} r'^l P_{l,-m}(\sin\phi') \exp(-im\lambda') \rho(r', \phi', \lambda') dV'$$

The K_{lm} coefficients depend on the shape and density function of the Earth. They are called harmonics of the geopotential (for *U*, and $\mathcal{R}$, are harmonic functions), of degree *l* and order *m*. In practice, noting that K_{lo} is real, we define real coefficients C_{lm}, S_{lm} for any $m > o$ by :

$$K_{lm} = \frac{1+\delta_{om}}{2}\left(C_{lm} - iS_{lm}\right)$$

$$K_{l,-m} = \frac{1+\delta_{om}}{2}\left(C_{lm} + iS_{lm}\right)\frac{(l+m)!}{(l-m)!}(-1)^m \tag{42}$$

(where $\delta_{om} = o$ if $m \neq o, \delta_{oo} = 1$). When $m = o$, $K_{lo} = C_{lo}$ and $S_{lo} = o$ and it is then easy to verify that $\mathcal{R}$ can be written as :

$$\mathrm{R} = \frac{GM}{r}\sum_{l=1}^{\infty}\left(\frac{R}{r}\right)^{l}\left[C_{lo}\,P_l\left(\sin\phi\right) + \sum_{m=1}^{l}\left(C_{lm}\cos m\lambda + S_{lm}\sin m\lambda\right)P_{lm}\left(\sin\phi\right)\right] \tag{43}$$

The C_{lo} coefficients are sometimes denoted as - J_l, and are called zonal harmonics, since they characterize variations of U which are independent of the longitude. The other harmonics $\left(C_{lm}, S_{lm}\right)$ are called tesseral ; a peculiar case is when $l = m$ and the $\left(C_{ll}, S_{ll}\right)$ are named sectorial harmonics. Practically, the origin of $\left(\Sigma_o\right)$ is taken at the Earth's center of mass and the Z axis along the mean Earth axis of rotation, assumed to be a principal axis of inertia. This hypothesis implies that $C_{10} = C_{11} = S_{11} = C_{21} = S_{21} = 0$. Furthermore, we have the important relations :

$$C_{20} = -\frac{1}{MR^2}\left(C - \frac{A+B}{2}\right)$$

$$C_{22} = \frac{1}{MR^2}\frac{B-A}{4}$$

where A,B,C are the moments of inertia of (E) in $\left(\Sigma_o\right)$.

In the following, equation (43) which gives the expression of $\mathcal{R}$ in terms of the spherical harmonics of the geopotential, will start at $l = 2$. It will also be used with normalized Legendre functions $\overline{P}_{lm}(x)$ and normalized harmonics $\left(\overline{C}_{lm}, \overline{S}_{lm}\right)$ such that :

$$\overline{P}_{lm}\cdot\left(\overline{C}_{lm}, \overline{S}_{lm}\right) = P_{lm}\cdot\left(C_{lm}, S_{lm}\right)$$

$$\overline{P}_{lm}(x) = \left[\left(2-\delta_{om}\right)(2l+1)\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_{lm}(x) = \nu_{lm}P_{lm}(x) \tag{44}$$

This normalization is such that :

$$\frac{1}{4\pi}\int\int_{unit\ sphere} \overline{P}_{lm}^{2}\left(\sin\phi\right)\begin{bmatrix}\cos^2 m\lambda \\ \sin^2 m\lambda\end{bmatrix}\cos\phi d\phi d\lambda = 1$$

Hence :

$$\begin{aligned}\mathrm{R} &= \frac{GM}{r}\sum_{l=2}^{\infty}\left(\frac{R}{r}\right)^{l}\sum_{m=o}^{l}\left(\overline{C}_{lm}\cos m\lambda + \overline{S}_{lm}\sin m\lambda\right)\overline{P}_{lm}\left(\sin\phi\right) \\ &= \frac{GM}{r}\sum_{m=o}^{\infty}\sum_{l=\sup(m,2)}^{\infty}\left(\frac{R}{r}\right)^{l}\left(\overline{C}_{lm}\cos m\lambda + \overline{S}_{lm}\sin m\lambda\right)\overline{P}_{lm}\left(\sin\phi\right) \\ &= \sum_{l,m}\mathrm{R}_{lm}\end{aligned}$$

2•2. THE REPRESENTATION OF THE GEOID SHAPE

The geoid is a conventional equipotential surface of the total potential $W = U + C$, where U is the gravitational potential, and C is the centrifugal potential of the rotating Earth ($C = \left[\dot{\theta}^2 r^2 \cos^2\phi\right]/2$ with $\dot{\theta}$ = sidereal rotation rate). This equipotential, in the oceanic areas, is the surface the sea would have if there was no motion of the sea water, even averaged over an infinite time (this assumes that mass movements, such as those due to tectonic motions or internal convection, are neglected in the "solid" Earth) ; this geoid physical definition is implicity extended (mathematically valid) over the continental areas. If the Earth was fluid and composed of (for instance) homogeneous confocal layers, its surface would be a perfect ellipsoid of revolution. Besides the observed fact that the Earth's surface may actually be approximated by such an ellipsoid flattened at the poles, this is why the shape of the geoid is described with respect to an ellipsoid of revolution, called a dynamical ellipsoid. It is defined as having the same mass, center of mass and mean rotation axis as the Earth's ; it has a prescribed semi-major axis a_e and a flattening $\alpha = \left(a_e - a_p\right)/a_e$ (a_p : semi-minor (polar) axis) ; it rotates with the Earth with the same sidereal rate $\dot{\theta}$ and its surface is an equipotential of its own total potential $W_E = U_E + C$, (U_E = gravitational part) ; conventionally, the value of W_E on its surface is taken equal to the value of the real potential W on the geoid surface.

Under these assumptions, the height, usually denoted by $\boldsymbol{N}$ in physical geodesy, of the geoid with respect to the ellipsoid, counted positively along the outward normal $\bar{n}$ to the ellipsoid is given by (Brun's formula) :

$$\boldsymbol{N} = \frac{W - W_E}{\gamma} = \frac{U - U_E}{\gamma} \tag{45}$$

with γ being the gravity on the ellipsoid : $\gamma = \left|\partial W_E / \partial n\right|$. As a result, and since the ellipsoid gravitational potential expansion involves even degree zonal terms only, we have :

$$\boldsymbol{N} = \frac{GM}{r\gamma}\sum_{l=2}^{\infty}\left(\frac{R}{r}\right)^{l}\sum_{m=o}^{l}\left(\bar{C}_{lm}^{*}\cos m\lambda + \bar{S}_{lm}\sin m\lambda\right)\bar{P}_{lm}\left(\sin\phi\right) \tag{46}$$

with :

$\bar{C}_{lm}^{*} = \bar{C}_{lo} - \bar{C}_{lo}\left(ellipsoid\right)\ldots$ if l is even

$= \bar{C}_{lo}\ldots$ if l is odd

$\bar{C}_{lm}^{*} = \bar{C}_{lm}$ if $m > 0$.

This expression is often used in the simplified form (taking $\gamma = GM/r^2$ and $r = R = a_e$ instead of their mathematical expressions at the surface of the reference ellipsoid) :

$$\boldsymbol{N} \approx R\sum_{l,m}\left(\bar{C}_{lm}^{*}\cos m\lambda + \bar{S}_{lm}\sin m\lambda\right)\bar{P}_{lm}\left(\sin\phi\right) \tag{47}$$

Figure 7 shows how the geoid is positioned with respect to the reference ellipsoid. It also shows other surfaces, close to it, which are relevant to satellite altimetry.

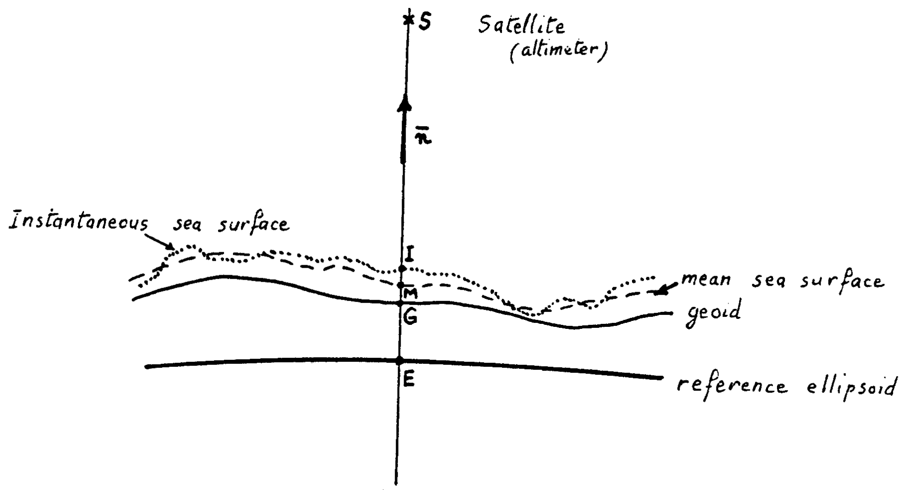

Fig. 7. Surfaces to be considered in satellite altimetry

$\boldsymbol{N} = \overline{EG}$ measures the departure of the geoid shape from the ellipsoid.

$\overline{GI}$ is the dynamic topography (instantaneous), $\overline{GM}$ its mean value.

An altimeter on board satellite S measures $\overline{SI}$.

If S is known from ground tracking observations and a posteriori orbit determination, then $\overline{ES}$ is known ; $\overline{SI}$ being measured, $\overline{EI}$ is known.

2•3. TRANSFORMATION IN ORBITAL ELEMENTS ; KAULA'S SOLUTION

Our goal is now to use Lagrange equations to derive the main geopotential perturbations on a satellite orbit. It is therefore necessary to transform $\mathcal{R}$ as given by (44) and expressed in (Σ_o), in a function of all six orbital elements. It is clear that r will involve the elements (a, e, M), whereas ϕ and λ will involve I, Ω, ω and M. The transformation is therefore splitted into two parts.

2•3.1- Transformation of $\overline{P}_{lm}(\sin\phi)\cos m\lambda$ *and* $\overline{P}_{lm}(\sin\phi)\sin m\lambda$

There are several ways of achieving it. One, originally due to Kaula (hence the name of Kaula's solution) starts from the exact expression of $\overline{P}_{lm}(\sin\phi)$ in terms of powers of $\sin\phi$, divided by $\cos^m\phi$, transforms $\cos m\lambda$ and $\sin m\lambda$ in terms of powers of $\cos(\omega+v), \sin(\omega+v), \cos I$, with the factor $\exp\left[im(\Omega-\theta)\right]/\cos^m\phi$. There remains a triple summation which gives the quantities in terms of cosines and sines of the argument $(l-2p)(\omega+v)+m(\Omega-\theta)$ with the so-called Kaula's inclination functions $\overline{F}_{lmp}(I)$ in factor (Kaula, 1966).

Another derivation starts from the theorem on the rotation of the spherical harmonic functions $Y_{lm}(\phi,\lambda) = P_{lm}(\sin\phi)\exp(im\lambda)$, when going from a reference system (σ) to another one (σ') by three rotations according to the three usual Euler angles Ψ,Θ,Φ ; this theorem states that :

$$(l-m)!Y_{lm}(\phi,\lambda)=\sum_{m'=-l}^{+l}(l-m')!Y_{lm'}(\phi',\lambda')E_{lm}^{m'}(\Psi,\Theta,\Phi) \qquad (48)$$

The Euler functions $E_{lm}^{m'}$ are defined as :

$$E_{lm}^{m'}(\Psi,\Theta,\Phi)=(-1)^{l-m}\exp\left[i(m'-m)\frac{\pi}{2}\right]\exp\left[i(m\Psi+m'\Phi)\right]C_{lm}^{m'}\left(\frac{\Theta}{2}\right)$$

where the $C_{lm}^{m'}$ are the Clifford trigonometric polynomials :

$$C_{lm}^{m'}\left(\frac{\Theta}{2}\right)=\sum_{j=j_{\inf}}^{j_{\sup}}(-1)^j\binom{l-m}{j}\binom{l+m}{m+m'+j}\cos^{v}\frac{\Theta}{2}\sin^{2l-v}\frac{\Theta}{2}$$

with :

$$j_{\inf}=\max(0,-m-m')$$
$$j_{\sup}=\min(l-m,l-m')$$
$$v=2j+m+m'$$

We apply this transformation to $\sigma=\left(\overline{X}_o,\overline{Y}_o,\overline{Z}_o\right)=\left(\Sigma_o\right)$ and $\sigma'=\left(\overline{r},\overline{s},\overline{w}\right)$ - fig. 8.

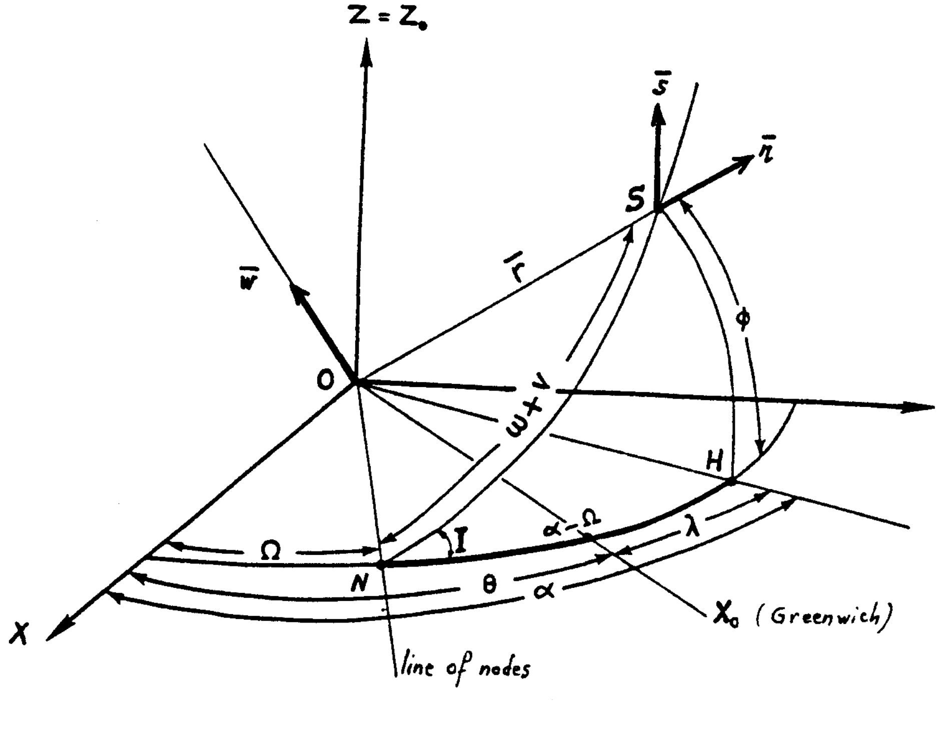

Fig. 8. Angles encountered in Kaula's transformation

Hence : $\Psi = \Omega - \theta, \Theta = I, \Phi = \omega + v$. In (σ'), we have $\lambda' = o, \phi' = o$. We take advantage of the fact that $P_{lm'}(o) = o$ if $l - m'$ is odd and $P_{lm'}(o) = (-1)^{[(l-m')/2]}$ $(l+m')! / \left\{2^l [(l-m')/2]! [(l+m')/2]!\right\}$ if $l - m'$ is even. The result is :

$$P_{lm}(\sin\phi)\exp(im\lambda) = i^{l-m} \sum_{p=o}^{l} D_{lmp}(I) \exp\left[i(l-2p)(\omega+v) + m(\Omega-\theta)\right] \quad (49)$$

with :

$$D_{lmp}(I) = (-1)^{l-m} \frac{(l+m)!}{2^l\, l!} \binom{l}{p} \sum_{j=j_1}^{j_2} (-1)^j \binom{2p}{j} \binom{2l-2p}{l-m-j} \left(\cos\frac{I}{2}\right)^{l+m-2p+2j} \left(\sin\frac{I}{2}\right)^{l-m+2p-2j}$$

(50)

$\left(j_1 = \max(o, 2p - l - m),\ j_2 = \min(l - m, 2p)\right)$.

These expressions are easier to evaluate than the Kaula's original ones. They are related to the classical functions $F_{lmp}(I)$ by $(m \geq o)$:

$$\begin{aligned} D_{lmp}(I) &= (-1)^{l-m+[(l-m)/2]} F_{lmp}(I) \\ D_{l,-m,p}(I) &= (-1)^{[(l-m)/2]} \frac{(l-m)!}{(l+m)} F_{l,m,l-p}(I) \end{aligned} \quad (51)$$

There exist numerous recursive relations between the F_{lmp}, or the D_{lmp} functions, which are more efficient for numerical evaluations, especially for large values of *l, m, p*.

2•3.2- Transformation of terms containing r and v

We take again formula (41) for $\mathcal{R}$ with $P_{lm}(\sin\phi)\exp(im\lambda)$ being replaced by (49). We have to transform $r^{-l-1}\exp\left[i(l-2p)(\omega+v)\right] = r^{-l-1}\exp\left[i(l-2p)v\right]\exp\left[i(l-2p)\omega\right]$. From the definition of the Hansen coefficients we immediately write :

$$\begin{aligned} \frac{1}{r^{l+1}}\exp\left[i(l-2p)v\right] &= \frac{1}{a^{l+1}} \sum_{k=-\infty}^{+\infty} X_k^{l-1,l-2p} \exp(ikM) \\ &= \sum_{q=-\infty}^{+\infty} X_{l-2p+q}^{-l-1,l-2p} \exp\left[i(l-2p+q)M\right] \end{aligned} \quad (52)$$

where the second expression is obtained by a change of index : $k = l - 2p + q$. Kaula introduced the notation :

$$G_{lpq}(e) = X_{l-2p+q}^{-l-1,l-2p} \tag{53}$$

From (22), it is clear that $G_{lpq}(e) = o\left(e^{|q|}\right)$. For most geodetic satellites, e is small $\left(< 10^{-2}\right)$, and only terms with $q = o, q = \pm 1$ and sometimes $q = \pm 2$ need to be taken into account in (52) for sufficient accuracy in the analytical solution.

Using (22) and (53), it is easy to get :

. if $q > o$:

$$G_{lpq} = \left(-\frac{e}{2}\right)^q \sum_{t=o}^{q} \binom{2p-2l}{q-t} \frac{(-1)^t}{t!} (l-2p+q)^t + o\left(e^{q+2}\right)$$

When $p < l$:

$$\binom{2p-2l}{q-t} = (-1)^{q-t} \binom{2l-2p+q-t-1}{q-t}$$

When $p = l$:

$$G_{llq} = \left(-\frac{e}{2}\right)^q \frac{(-1)^q}{q!} (q-l)^q + o\left(e^{q+2}\right)$$

. if $q < o$:

$$G_{lpq} = \left(-\frac{e}{2}\right)^{-q} \sum_{t=o}^{-q} \binom{-2p}{-q-t} \frac{(l-2p+q)^t}{t!} + o\left(e^{-q+2}\right)$$

When $p > o$:

$$\binom{-2p}{-q-t} = (-1)^{-q-t} \binom{2p-q-t-1}{-q-t}$$

When $p = o$:

$$G_{loq} = \left(-\frac{e}{2}\right)^{-q} \frac{(q+l)^{-q}}{(-q)!} + o\left(e^{-q+2}\right)$$

. if $q = o, G_{lpo} = 1 + o\left(e^2\right)$. One sometimes needs the term in e^2 ; it is computed as $(t = 1, s = o$ in formula (22), and $n+m+1 = -2p, n-m+1 = 2p-2l, m = l-2p)$:

$$\sum_{j=o}^{1}\sum_{h=o}^{j}\binom{-2p}{j-h}\frac{(l-2p)^h}{h!}\sum_{k=o}^{j}\binom{2p-2l}{j-k}\frac{(l-2p)^k}{k!}(-1)^k$$
$$\left[2\binom{l+1-h-k}{1-j}-\binom{l+2-h-k}{1-j}\right]\frac{e^2}{4}=\left[l+(4p-3l)(l-4p)\right]\frac{e^2}{4}$$

To summarize, we have :

$$G_{lpo} = 1+g_{lpo}\frac{e^2}{2}+o\left(e^4\right)$$
$$G_{lp\pm1} = g_{lp\pm1}\,e+o\left(e^3\right) \qquad (54)$$
$$G_{lp\pm2} = g_{lp\pm2}\frac{e^2}{2}+o\left(e^4\right)$$

so that :

$$G'_{lpo} = g_{lpo}\,e+o\left(e^3\right)$$
$$G'_{lp\pm1} = g_{lp\pm1}+o\left(e^2\right) \qquad (55)$$
$$G'_{lp\pm2} = g_{lp\pm2}\,e+o\left(e^3\right)$$

where :

$$g_{lpo} = \left[l+(4p-3l)(l-4p)\right]/2$$
$$g_{lp1} = (3l-4p+1)/2$$
$$g_{lp-1} = (4p-l+1)/2 \qquad (56)$$
$$g_{lp2} = (l-p)(2l-3p+5/2)+(l-2p+2)^2/4$$
$$g_{lp-2} = p(3p-l+5/2)+(l-2p-2)^2/4$$

2•3.3- Final form of the geopotential disturbing function

Putting together (49) and (52), we obtain :

$$\mathcal{R} = \frac{\mu}{a} \sum_{l=2}^{\infty} \left(\frac{R}{a} \right)^{l} \sum_{m=-l}^{+l} i^{\,l-m} K_{lm} \sum_{p=o}^{l} D_{lmp}(I) \sum_{q=-\infty}^{+\infty} G_{lpq}(e) \exp i \psi_{lmpq} \tag{57}$$

where :

$$\psi_{lmpq} = (l-2p)\,\omega + (l-2p+q)\,M + m(\Omega - \theta) \tag{58}$$

Actually $\mathcal{R}$ is a real function and, taking account of (42) and (51), adopting normalized coefficients $\left(\overline{C}_{lm}, \overline{S}_{lm}\right)$, we find :

$$\begin{aligned} \mathcal{R} &= \frac{\mu}{a} \sum_{l=2}^{\infty} \left(\frac{R}{a} \right)^{l} \sum_{m=o}^{l} \sum_{p=o}^{l} \overline{F}_{lmp}(I) \sum_{q=-\infty}^{+\infty} G_{lpq}(e)\, S_{lmpq}\,(\Omega, \omega, M, \theta) \\ &= \sum_{l,m} \mathcal{R}_{lm} = \sum_{l,m,p,q} \mathcal{R}_{lmpq} \end{aligned} \tag{59}$$

with :

$$\overline{F}_{lmp}(I) = \nu_{lm} F_{lmp}(I)$$

(cf. formula (44)).

$$S_{lmpq} = \widetilde{C}_{lm} \cos \psi_{lmpq} + \widetilde{S}_{lm} \sin \psi_{lmpq}$$

and :

$\widetilde{C}_{lm} = \overline{C}_{lm}$, $\widetilde{S}_{lm} = \overline{S}_{lm}$ if $l-m$ is even,

$\widetilde{C}_{lm} = -\overline{S}_{lm}$, $\widetilde{S}_{lm} = \overline{C}_{lm}$ if $l-m$ is odd .

2•3.4- First order perturbations in the elements

The term of the geopotential which dominates the disturbing function is R_{20} since $\overline{C}_{20}$ $\left(\text{or } -\overline{J}_2\right)$ is at least hundred times larger than any other $\overline{C}_{lm}$ or $\overline{S}_{lm}$ (it has been found empirically that the magnitude of these coefficients decrease approximately as $10^{-5}/l^2$ - Kaula's rule). We can therefore get a fairly good estimation of the major perturbations by restricting ourselves to :

$$\mathrm{R}_{20} = \frac{\mu}{a} C_{20} \left(\frac{R}{a}\right)^2 \sum_{p,q} F_{20p}(I)\, G_{2pq}(e) \cos\left[(2-2p)\,\omega + (2-2p+q)\, M\right],$$

(written here in non normalized form with $C_{20} \approx 1.08262810^{-3}$).

Following the successive approximation technique described in 1.4, we first get the (main) secular terms on Ω, ω, M from the above, with $p = 1$ and $q = o$, that is :

$$\mathrm{R}_{2010} = \frac{\mu}{a} C_{20} \left(\frac{R}{a}\right)^2 F_{201}(I)\, G_{210}(e)$$

where :

$$F_{201}(I) = 3/4 \sin^2 I - 1/2$$

$$G_{210}(e) = \left(1-e^2\right)^{-3/2}, \text{ exactly.}$$

From the last three Lagrange equations, we find :

$$\dot{\Omega}_{\sec} = n_{\Omega} = \frac{3}{2} n C_{20} \left(\frac{R}{a}\right)^2 \frac{1}{\left(1-e^2\right)^2} \cos I$$

$$\dot{\omega}_{\sec} = n_{\omega} = \frac{3}{4} n C_{20} \left(\frac{R}{a}\right)^2 \frac{1}{\left(1-e^2\right)^2} \left(1 - 5\cos^2 I\right) \qquad (60)$$

$$\dot{M}_{\sec} = n_M = n - \frac{3}{4} n C_{20} \left(\frac{R}{a}\right)^2 \frac{1}{\left(1-e^2\right)^{3/2}} \left(3\cos^2 I - 1\right)$$

These are very important formulas. They show that the mean orbital plane has a precession motion which is retrograde if $0 \le I \le 90$ (since $C_{20} < 0$) and prograde if $90° < I \le 180°$ (at this degree of approximation, it is extremely small for $I = 90°$) ; the line of apsides rotates (motion of ω) in this orbital plane clockwise if $I > I_c$, counterclockwise if $I < I_c$, with $I_c \approx 63°26'$; I_c is the critical inclination and the periapsis undergoes a peculiar libration motion when $I = I_c$ (in what follows, we will always assume that $I \neq I_c$) ; finally, with respect to the mean motion *n* (in the absence of perturbations), the satellite goes faster on its orbit if $I < I_o$, and slower if $I > I_o$ with $3\cos^2 I_o - 1 = o \quad \left(I_o \approx 35°16'\right)$.

In reality, it is easy to see that other terms, namely the zonal harmonics $C_{2k,o}$ (of even degree), for $k > 2$, also give secular perturbations which can be computed as above in a first approximation ; in the following, we will assume that $\dot{\Omega}_{\sec}, \dot{\omega}_{\sec}, \dot{M}_{\sec}$ (denoted simply $\dot{\overline{\Omega}}, \dot{\overline{\omega}}, \dot{\overline{M}}$) contain these perturbations.

One must realize that there is no mean to have secular perturbations on *a, e, I* with this type of disturbing function.

To finish with, we apply the remaining of the procedure described in 1.4, and we obtain (including the variations in *M* resulting from changes in the mean motion *n*, arising from perturbations of the semi-major axis) :

$$\Delta\alpha = \sum_{lmpq} \Delta\alpha_{lmpq} \tag{61}$$

where α represents anyone of the orbital elements, and the (*lmpq*) set of indices is such that it does not produce any secular effect (already included in $\dot{\Omega}, \dot{\omega}, \dot{M} = n$), that is $\dot{\psi}_{lmpq} \neq o$.

The $\Delta\alpha_{lmpq}$ for the metric elements *a, e, I* are of the form :

$$\Delta\alpha_{lmpq} = C^{\alpha}_{lmpq}\left(\overline{a}, \overline{e}, \overline{I}, \dot{\overline{\Omega}}, \dot{\overline{\omega}}, \dot{\overline{M}}\right) S_{lmpq}\left(\Omega, \omega, M, \theta\right) \tag{62}$$

and for the angular elements Ω, ω, M :

$$\Delta\alpha_{lmpq} = C^{\alpha}_{lmpq}\left(\overline{a}, \overline{e}, \overline{I}, \dot{\overline{\Omega}}, \dot{\overline{\omega}}, \dot{\overline{M}}\right) S^{*}_{lmpq}\left(\Omega, \omega, M, \theta\right) \tag{63}$$

where : $\overline{a}, \overline{e}, \overline{I}$ are the mean values of *a, e, I,* as opposed to their osculating values $a = \overline{a} + \sum\left(\Delta a_{lmpq}\right)$, etc ..., and $\dot{\overline{\Omega}}, \dot{\overline{\omega}}, \dot{\overline{M}}$ are the mean rates of Ω, ω, M evaluated with the secular terms as said above ; that is for instance, $\Omega\left(osculating\right) = \Omega_o + \dot{\overline{\Omega}}\left(t - t_o\right) + \sum \Delta\Omega_{lmpq}$.

S_{lmpq} is as in (59), and $S^*_{lmpq} = \widetilde{C}_{lm} \sin\psi_{lmpq} - \widetilde{S}_{lm} \cos\psi_{lmpq}$. ψ_{lmpq} itself is evaluated with the mean angular elements $\Omega_o + \dot{\overline{\Omega}}(t - t_o)$, $\omega_o + \dot{\overline{\omega}}(t - t_o)$, $M_o + \overline{n}(t - t_o)$ and with $\theta = \theta_o + \dot{\theta}(t - t_o)$. In the remaining of this course, we will drop all the overbars to simplify the notations since there should be no confusion : first order perturbations are evaluated with the values of the mean elements.

The C^{α}_{lmpq} coefficients are the following :

$$
\begin{aligned}
C^{a}_{lmpq} &= 2AG_{lpq}(l - 2p + q) \Big/ \dot{\psi}_{lmpq} \\
C^{e}_{lmpq} &= \frac{A}{a}\frac{\sqrt{1-e^2}}{e} G_{lpq} \left[\sqrt{1-e^2}\,(l - 2p + q) - (l - 2p)\right] \Big/ \dot{\psi}_{lmpq} \\
C^{i}_{lmpq} &= \frac{A}{a}\frac{1}{\sin I\sqrt{1-e^2}} G_{lpq} \left[(l - 2p)\cos I - m\right] \Big/ \dot{\psi}_{lmpq} \qquad (64) \\
C^{\Omega}_{lmpq} &= \frac{A'}{a}\frac{1}{\sin I\sqrt{1-e^2}} G_{lpq} \Big/ \dot{\psi}_{lmpq} \\
C^{\omega}_{lmpq} &= \frac{1}{a}\left[\frac{\sqrt{1-e^2}}{e} AG'_{lpq} - \frac{\cos I}{\sin I\sqrt{1-e^2}} A'G_{lpq}\right] \Big/ \dot{\psi}_{lmpq} \\
C^{M}_{lmpq} &= \frac{A}{a}\left[2(l+1)G_{lpq} - \frac{1-e^2}{e} G'_{lpq} - 3G_{lpq}\frac{(l - 2p + q)n}{\dot{\psi}_{lmpq}}\right] \Big/ \dot{\psi}_{lmpq}
\end{aligned}
$$

with :

$$A = na\left(\frac{R}{a}\right)^l \overline{F}_{lmp}(I)$$

$$A' = na\left(\frac{R}{a}\right)^l \overline{F}'_{lmp}(I)$$

As an example, we have computed the perturbations for the TOPEX-POSEIDON satellite with the following mean elements $a = 7714410$ m, $e = 9.3\,10^{-5}$, $I = 66°02$. They have been converted to rectangular coordinate perturbations in the Gauss system by the method which is the subject of chapter 3 for all *l, m, p, q's.*

R. Rapp's 1991 global geopotential model truncated at degree and order 60 has been used, and $|q|$ limited at 2. Then, since the perturbations for given $(\overline{C}_{lm}, \overline{S}_{lm})$ are composed of many frequencies, the r.m.s. has been computed. The diagram on figure 9 shows the r.m.s. perturbation in position, in meters, for each couple of harmonics (for low degrees and orders, the perturbations are quite large and their graphic representation was truncated ...).

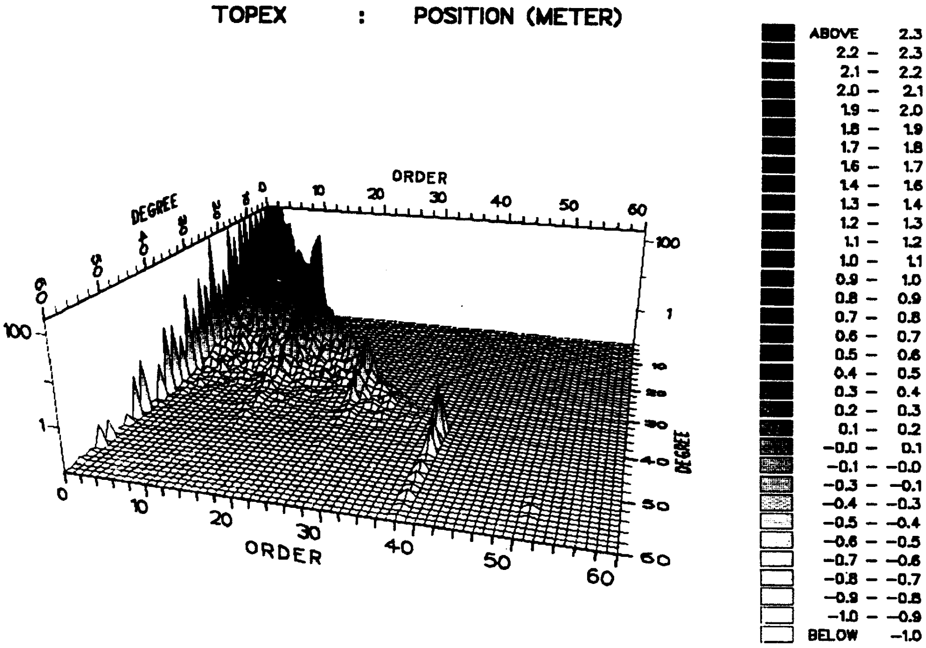

Fig. 9. Diagram of r.m.s. perturbations in the position of the Topex-Poseidon satellite.

2•3.5- Choosing the orbit of a satellite

It all depends on the usage of the satellite, of the on board sensors and their operational constraints.

The mean motion is quite important for it is the major angular parameter which very directly interacts with the sidereal time rate $\dot{\theta}$ and it conditions greatly the overall coverage. The mean semi-major axis which corresponds to it immediately places the spacecraft far enough from the Earth's upper atmosphere or directly in it (e.g. from 200 to 1 000 km) which may entail problems as concerns the mission life-time, the proper operation of some sensors, the attitude and orbit controls of the satellite ... ; also, one must note the decrease of the geopotential perturbations as $(R/a)^l$, (apart from sharp resonance cases), of which one may take advantage, for instance in the case of geodynamic satellites (e.g. LAGEOS). The mean eccentricity will usually be rather small, so as to operate at more or less constant altitude, apart from variations due to the radial orbit perturbations and due to the Earth's flattening. The inclination is a very important parameter since it is through it that the orbital plane precesses and, for many sensors of geodetic and Earth observation missions, it governs the coverage one finally obtains thoughout the mission.

Important cases are : the polar inclination by which the orbital plane is practically fixed in space (if the altitude is sufficient to neglect the effects of drag) ; the heliosynchronous case in which the orbit plane follows (approximately) the motion of the sun with respect to Earth, that is $\dot{\Omega} = 360°/365.2422\, d = 0.98565°/\text{day}$ (it cannot follow the sun exactly since the right ascension of the sun does not vary linearly but has additional periodic terms which depend on the Earth mean anomaly, eccentricity and obliquity), which requires an inclination generally in the range of 96° to 100° : $I(\text{helios.}) = \cos^{-1}\left[-4.784204.10^{-15} a^{7/2}\right]$, with a in km.

In all cases, figure 10 illustrates how successive tracks are placed with respect to the Earth, from which one can derive algorithms to compute the coverage of the ground tracks or to determine repeat orbits. The algorithms are based on the value of the longitude interval, $\Delta\lambda$, between two successive tracks, with respect to Earth.

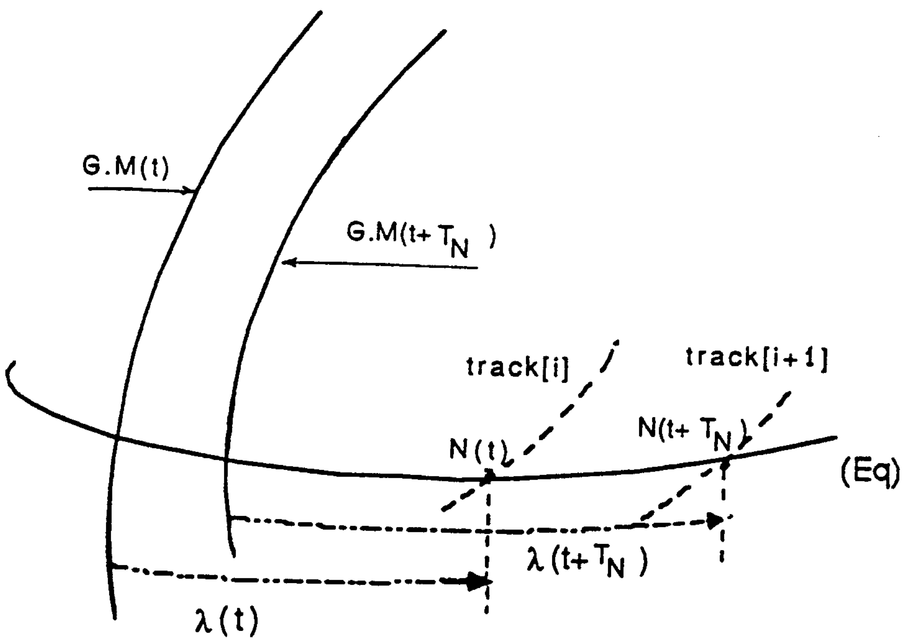

Fig. 10. Geometry of successive tracks

N : ascending node

T_N : nodal period

G.M. : Greenwich meridian

Eq : Earth equator

We have :

$$\Delta\lambda = \lambda\left(t + T_N\right) - \lambda(t) ,$$
$$= \left(\dot{\Omega} - \dot{\theta}\right) T_N$$

where $\dot{\Omega}$ is the secular drift of the ascending node and T_N the nodal (draconitic period), given by :

$$T_N = 2\pi / \left(\dot{M} + \dot{\omega}\right)$$

with, as before :

$\dot{M}$ = "mean" mean motion $\left(\dot{M} > o\right)$

$\dot{\omega}$ = secular drift of argument of periapsis $\left(|\dot{\omega}| \ll \dot{M}\right)$

$\dot{\theta}$ = sidereal time rate

Two types of problems can then be solved :

(a) Resolution at the equator versus time :

Let us call ρ_j the resolution on the equator after a time interval ΔT_j counted from the beginning of the mission (with $\Delta T_j < \Delta T_{j+1} \ldots$), that is after an integer number of nodal revolutions, K_j.

We have :

$\Delta T_j = K_j T_N$

and we write :

$\rho_j = \overline{R}\Delta_j$

where $\overline{R}$ is the (mean) Earth equatorial radius.

At the beginning, we have :

$\rho_o = 2\pi\overline{R}$ and $K_o = 1$

Then the series $\left\{\rho_j, \Delta T_j\right\}_j$ is given by the following sequence :

- let us define :

. if $j = 1$: $A_1 = 2\pi, \quad \Delta_1 = |\Delta\lambda|$

if $j > 1$: $B_j = q_{j-1}\,\Delta_{j-1} - A_{j-1}$

$$A_j = \Delta_{j-1}$$

$$\Delta_j = \inf\left(B_j, A_j - B_j\right)$$

- then, for all $j \geq 1$: $q_j = \left[A_j / \Delta_j + 1\right]$,[...] = integer part

$$K_j = \left[2\pi / \Delta_j + 1/2\right]$$

$$\rho_j = \overline{R}\,\Delta_j$$

$$\Delta T_j = K_j T_N$$

This algorithm takes account only of ascending <u>or</u> descending passes. If both types of passes are considered, which is reasonable if the orbital eccentricity is small - that is the spacecraft altitude will be almost the same at the descending and ascending nodes and sensor “operation“ conditions may be similar too, the actual resolution will be between $\rho_i/2$ and ρ_i.

Finally, if one is interested in the mean resolution at some latitude, resolution numbers must be multiplied by the cosine of that angle.

(b) Determination of repeat orbits of given repeat period :

A repeat orbit is characterized by the existence of integer solutions $\{h,k\}, h \in \mathbf{N}, k \in \mathbf{Z}$, to the following equations :

$$hT_N = T_{rep}$$

$$h(\dot{\Omega}-\dot{\theta})T_N = 2k\pi, \text{ or } \frac{\dot{\omega}+\dot{M}}{\dot{\Omega}-\dot{\theta}} = \frac{h}{k} \qquad (65)$$

where notations are as before and the given repeat period is T_{rep}. Being given a_o, e_o, I_o (usual metric elements), and allowed intervals of their variations : $[a_o-\Delta a, a_o+\Delta a] = \mathrm{A}$, $[e_o-\Delta e, e_o+\Delta e] = \mathrm{E}$, $[I_o-\Delta I, I_o+\Delta I] = \mathrm{J}$, one searches the possible values $h_1, h_2 \ldots h_p$ and associated values (*k*) which may satisfy the equations for $a \in \mathrm{A}, e \in \mathrm{E}, I \in \mathrm{J}$. Actually, the h_i's are all consecutive, that is $h_{\min} \le h \le h_{\max}$ (and $h_i = h_{\min} + i - 1$) and, for a given *h*, possible values of *k* are found to be between $k_{\min}(h)$ and $k_{\max}(h)$. There may be no such value for a given T_{rep}.

For any couple of values (*h,k*) and a given value of *e* in *E*, one then tries to find *a* and *I* so that $\dot{\omega}(a,e,I), \dot{M}(a,e,I), \dot{\Omega} = (2\pi/T_N)(k/h) + \dot{\theta}$ satisfy exactly the system.

There may be no solution, or sometimes solutions outside *A* and *J*.

The physical interpretation of the repeat orbit is that the ground track repeats itself after *h* (nodal) revolutions of the satellite in the orbital plane and after *k* revolutions of the orbital plane itself about the Earth's mean rotation axis and with respect to the Earth's surface : $T_{rep} = hT_N = kT_{\Omega-\theta}$.

In reality nearly every circular trajectory resembles a repeating one since any real value of $(\dot{\omega}+\dot{M})/(\dot{\Omega}-\dot{\theta})$ may be approximated by a ratio of two integers. A practical problem might be that the integer values become quite large for an accurate approximation of this ratio ; therefore one usually limits oneself to repeat periods which are less than a few months or so.

The longitude spacing of the ground tracks is obviously 360°/h. For example, SEASAT had in its last month a repeat orbit at the mean altitude of 790 km with I = 108°, resulting in $h/k = -43/3$, hence a longitude spacing of the ground tracks at the equator equal to 360°/43 = 8.37°, TOPEX-POSEIDON, with a = 7714.5 km, e = 9.5 10^{-5} and I = 66.039°, is such that $(\dot{\omega} + \dot{M})/(\dot{\Omega} - \dot{\theta}) = -12.7 = -127/10$, hence a longitude spacing of 2.83° for a repeat period of 9.92 days.

It is interesting to look at the spectral characteristics of repeat arc differences in this case. All orbital elements being expressed, as in (62) and (63), as Fourier series with coefficients which are functions of the mean (fixed) metric elements, $\alpha\left(t + T_{rep}\right) - \alpha(t)$ is the product of such a coefficient (independent of t) by a sine or cosine of :

$$\psi_{lmpq}\left(t + T_{rep}\right) - \psi_{lmpq}(t)$$

Writing $\psi_{lmpq} = (l - 2p + q)(\omega + M) + m(\Omega - \theta) - q\omega$, taking account of $T_{rep} = h2\pi/(\dot{\omega} + \dot{M}) = k2\pi/(\dot{\Omega} - \dot{\theta})$, and then of $\Omega\left(t + T_{rep}\right) = \Omega(t) + \dot{\Omega}T_{rep}$, $\omega\left(t + T_{rep}\right) = \omega(t) + \dot{\omega}T_{rep}$, $M\left(t + T_{rep}\right) = M(t) + \dot{M}T_{rep}$, we find :

$$\psi_{lmpq}\left(t + T_{rep}\right) - \psi_{lmpq}(t) = \left[(l - 2p + q)h + mk\right]2\pi - q\dot{\omega}T_{rep}$$

To the order o in eccentricity $(q = o)$, we find that the argument after T_{rep} differs by a multiple of 2π ; therefore the differences of any two elements are equal to zero. In particular, the radial perturbations are the same on any ascending <u>or</u> descending arc ... but not necessarily at a cross-over between an ascending <u>and</u> a descending arc (it can be shown how they actually differ - see, for instance, Balmino, 1993). The term $q\dot{\omega}T_{rep}$ causes this result to be approximate : we can only say that all short periodic perturbations due to geopotential model errors are eliminated in repeat arc differences. In the case of a frozen repeat orbit, we have $\dot{\omega} \approx o$ and we can expect the effect of $q\dot{\omega}T_{rep}$ to be negligible.

2•4. THE DETERMINATION OF A GEOPOTENTIAL MODEL-OVERVIEW

Global modeling of the Earth's gravity field has been a concern since the beginning of the artificial satellites era. Observing the trajectories in space of such proof-masses allows in principle to determine the forces which act upon them and compute the coefficients inherent to their parameterization ; this is the oldest inverse problem of celestial mechanics. In practice, however, trajectories are observed from ground stations (sometimes from another satellite) by means of ranging devices (radars, laser system which now reach centimeter precision), range-rate measurement apparatus (measuring the Doppler effect), even tracking cameras which observed, in the old days, the directions to the satellites on the sky background. All these instruments have limitations (biases and noise) and, since satellites must be flown at a minimum altitude *H* if we want to live long enough (say above 350 km for a life-time of a few months - without manoeuvering the orbit), the attenuation factor $\left[R/(R+H)\right]^{l}$ ultimately limits the degree *l* (and order $m \le l$) to which we can determine the spherical harmonics of the geopotential.

Another important fact lies in the frequency spectrum of the geopotential orbital perturbations, which comes from the decomposition of the disturbing function $\mathcal{R}$ as given by (59). We can re-arrange the quadruple summation over (l, m, p, q) as follows : for a model to be determined up to degree and order *L*, we first write that :

$$\sum_{l=2}^{L}\sum_{m=o}^{L}\cdots=\sum_{m=o}^{L}\sum_{l=\max(m,2)}^{L}\cdots$$

Then, changing *p* into $k = l - 2p$, *q* into $s = l - 2p + q$, interchanging the summations over *l* and *k* and finally limiting the series in $G_{lpq}(e)$ to $|q| \le Q$, we readily find :

$$\mathrm{R}=\frac{\mu}{a}\sum_{m=o}^{L}\sum_{k=-L}^{+L}\sum_{s=k-Q}^{k+Q}\left\{\sum_{\substack{l=\max(m,2,|k|)\\ l-k:even}}^{L} i^{l-m}\,\overline{K}_{lm}\left(\frac{R}{a}\right)^{l}\overline{D}_{l,m,(l-k)/2}(I)\,G_{l,(l-k)/2,s-k}(e)\right\}\exp\left(i\,\psi_{ksm}\right) \quad (66)$$

with : $\psi_{ksm} = k\omega + sM + m(\Omega - \theta)$, and $\overline{D}_{lmp}$ being the normalized inclination function.

From (66) it is obvious that several harmonics (an infinity when $L \to \infty$) give rise to perturbations of the same frequency. By varying *s*, one goes from the so-called m-daily perturbations (period $\approx 2\pi/(m\dot{\theta})$ when $s = o$, with $\dot{\omega}$ and $\dot{\Omega} << \dot{\theta}$) to short period perturbations $(s \neq o)$ which all involve the same $\overline{K}_{lm}$ harmonics. For a given distribution of tracking stations it is usually not possible to observe well enough the orbit, that is to sample well enough the perturbations it undergoes, and the resulting observation equations are then insufficient to separate the different harmonics. That is why, with this approach of geopotential determination, it is necessary to have several satellites with varied altitudes and especially inclinations so as to get very different coefficients in the bracketted term of (66), hence independent observation equations for the harmonics. A very favourable situation is also when the orbit is in shallow resonance, that is when there exist (*l, m, p, q*) or equivalently (*k, s, m*) sets of indices such that $\dot{\psi}_{ksm} << n$ ($\dot{\psi}_{ksm}$ may eventually come to zero in cases of sharp resonance, but these are transient phenomena). Neglecting $\dot{\omega}$ and $\dot{\Omega}$ with respect to n and $\dot{\theta}$, such a situation occurs when $sn \approx m\dot{\theta}.$ If *n* is expressed in revolution per day, we have approximately $n \approx m/s.$ When *n* is an integer, the main resonant perturbations are with coefficients $\overline{K}_{ln}$ $(s = 1)$, then with $\overline{K}_{l,2n}(s = 2)$, $\overline{K}_{l,3n}(s = 3),\cdots$; if $n = r/2$ (r : integer), resonance occurs with $\overline{K}_{lr}(s = 2)$, $\overline{K}_{l,2r}(s = 4)$, and so on ... These enhanced perturbations allow to better determine the corresponding class (es) of harmonics. Finally, we remark that, if we have a polar satellite mission which results, after some time, in a ground track pattern with equatorial inter-track distance *Δ*, and if observations are made along the orbit at least every $(\Delta/R)/n$ seconds (*n* is here in rd/sec), then the data sample allows, in the perfect case, to recover all harmonics up to degree and order $L \approx [\pi R/(2\Delta)]$.

Most accurate geopotential models have also been determined by combining satellite data (from which satellite only solutions may be computed) with surface gravity measurements and also satellite derived geoid heights from past altimetry missions (Geos 3, Seasat, Geosat, ERS1-2, TOPEX-POSEIDON) - after correction for sea surface topography (the difference between the ocean surface and the geoid), from a model, or by simultaneously determining it. Equation (46) is the basis for performing this combination as far as the geoid height is concerned. A similar equation exists for the gravity anomalies Δg derived from surface measurements : $\Delta g \approx g_{measured}$ (reduced on the geoid) $- \gamma_{ref.ellipsoid}$ (on the ellipsoid) is the basic quantity used in this case (γ is the theoretical gravity - it is called "normal gravity") ; the relationship between Δg and the geopotential harmonics is :

$$\Delta g = \frac{\mu}{r^2} \sum_{l=2}^{\infty} (l-1) \left(\frac{R}{r}\right)^l \sum_{m=o}^{l} \left(\overline{C}_{lm}^* \cos m\lambda + \overline{S}_{lm} \sin m\lambda\right) \overline{P}_{lm}(\sin\phi) \quad (67)$$

where $\overline{C}_{lm}^*$ is as in (46). Such equations are in general rewritten for mean values which are derived from real measurements (or sometimes from predicted anomalies - with a larger uncertainty, to avoid artefacts in the poorly covered areas).

As an example of what can be obtained from the above described techniques, figures 11 and 12 show contour maps of the geoid height and predicted errors of one GRIM 4 (combined) solution (Schwintzer et al., 1997). Usual features in the geoidal surface are visible, and the errors accumulate over land areas not well covered by gravity data (altimetry data were used over the oceans, thus providing a much better control).

Fig. 11.

Fig. 12.

3• AN APPLICATION : THE RADIAL PERTURBATIONS DUE TO THE GEOPOTENTIAL

The radial perturbation, Δr, on a satellite orbit due to the geopotential may be derived in various ways, and with various approximations. One approach uses the Hill equations (Schrama, 1989), others start from the Lagrange equations and Kaula's formulation of the solution (Wagner, 1985 ; Rosborough, 1986, Engelis, 1987). Here, we will also start from the analytical expressions of the orbital perturbations derived from the Lagrange equations (formulas 64) and from $r(a,e,M)$ as given by (13) or (17). We will first derive the perturbations of order *o* and order 1 in an elementary fashion, to show some of the traps which are encountered when working with almost singular elements ($e \approx o$ usually), also to correct some mistakes made in earlier works in the terms of order 1.

We will therefore write :

$$\Delta r = \sum_{l,m,p} \left(\Delta r^{o}_{lmp} + \Delta r^{1}_{lmp} + \cdots \Delta r^{k}_{lmp} \cdots \right) \tag{68}$$

where the superscript stands for the order in eccentricity.

Perturbations in the two other orthogonal directions (transverse and normal) may be derived in a similar way. The transverse component is $\Delta\tau$, derived from $\Delta\tau = r\Delta\psi = r(\Delta v + \Delta\omega + \Delta\Omega \cos I)$; it can be evaluated from (18) and from the perturbations in $M, \omega,$ and Ω. The normal component, $\Delta\zeta$, comes from $\Delta\zeta = r\left[\Delta I \sin(\omega + v) - \Delta\Omega \sin I \cos(\omega + v)\right]$, requires to expand cos *v* and sin *v* by (19) and (20). Perturbations in all three directions will be given in the conclusion for completeness.

3•1. RADIAL PERTURBATIONS OF ZERO AND FIRST ORDER IN ECCENTRICITY

We start from (13) truncated at e^2 : $r = a\left[1 - e\cos M - e^2\ (\cos 2M - 1)/2\right]$ from which, by differentiation, we obtain :

$$\Delta r = (1 - e\cos M)\Delta a + a(e - \cos M - e\cos 2M)\Delta e + a\sin Me\Delta M$$

To obtain Δr^o and Δr^1, we rewrite Δr as :

$\Delta r =$	Δa ....with terms of order	0,1 in e
	$-e\Delta a\cos M$	0
	$-a\Delta e\cos M$	0,1
	$+ea\Delta e$	0
	$-ea\Delta e\cos 2M$	0

$+ ae\Delta M \sin M$ ……　　0,1

Now we use (64). We take advantage of (54), (55) and (56) and we keep terms with :

. $q = o, +1, -1$ for Δa

. $q = o, +1, -1$ for $e\Delta M$ and retain g_{lpo} for G'_{lpo} (and $1-e^2 \approx 1$)

. $q = o, \pm 1, \pm 2$ for $a\Delta e$ expanding $\sqrt{1-e^2}\left(l-2p+q\right)-\left(l-2p\right)$as $\approx -\left(l-2p\right)\frac{e^2}{2}+q$; therefore the singularity disappears when $q = o$; also, there is no singularity in e for $q = \pm 1$ due to $eg_{lp\pm 1}$; finally the remaining $\sqrt{1-e^2}$ term is replaced by 1, and $q = \pm 2$ generates a term in e.

Hence, dropping other e^2 terms which are generated by these choices of q and following the prescribed orders per term in Δr :

$$\Delta a = 2A\left\{\frac{l-2p}{\dot{\psi}_{lmpo}}S_{lmpo} + e\left[\frac{l-2p+1}{\dot{\psi}_{lmp1}}g_{lp1}\,S_{lmp1} + \frac{l-2p-1}{\dot{\psi}_{lmp-1}}g_{lp-1}\,S_{lmp-1}\right]\right\}$$

$$-e\Delta a\cos M = -2A\left\{\quad e\left[\frac{l-2p}{\dot{\psi}_{lmpo}}S_{lmpo}\cos M\right]\quad\right\}$$

$$-a\Delta e\cos M = -A\left\{\frac{g_{lp1}}{\dot{\psi}_{lmp1}}S_{lmp1}\cos M - \frac{g_{lp-1}}{\dot{\psi}_{lmp-1}}S_{lmp-1}\cos M\right.$$
$$\left. + e\left[-\frac{l-2p}{2\dot{\psi}_{lmpo}}S_{lmpo}\cos M + \frac{g_{lp2}}{\dot{\psi}_{lmp2}}S_{lmp2}\cos M + \frac{g_{lp-2}}{\dot{\psi}_{lmp-2}}S_{lmp-2}\cos M\right]\right\}$$

$$ea\Delta e = \left\{\quad e\left[\frac{g_{lp1}}{\dot{\psi}_{lmp1}}S_{lmp1} - \frac{g_{lp-1}}{\dot{\psi}_{lmp-1}}S_{lmp-1}\right]\quad\right\}$$

$$-ea\Delta e\cos 2M = -A\left\{\quad e\left[\frac{g_{lp1}}{\dot{\psi}_{lmp1}}S_{lmp1}\cos 2M - \frac{g_{lp-1}}{\dot{\psi}_{lmp-1}}S_{lmp-1}\cos 2M\right]\quad\right\}$$

$$ae\Delta M\sin M = A\left\{\quad -\frac{g_{lp1}}{\dot{\psi}_{lmp1}}S^*_{lmp1}\sin M - \frac{g_{lp-1}}{\dot{\psi}_{lmp-1}}S^*_{lmp-1}\sin M\right.$$
$$\left. + e\left[\left(2(l+1) - g_{lpo} - 3\left(l-2p\right)\frac{n}{\dot{\psi}_{lmpo}}\right)\frac{1}{\dot{\psi}_{lmpo}}S^*_{lmpo}\sin M\right]\right\}$$

Now, recalling the expressions of S_{lmpq} and S^{*}_{lmpq}, and noting that $\psi_{lmpq} \pm sM = \psi_{lmp,q\pm s}$ we form :

$$S_{lmpq} \cos sM = \frac{1}{2}\left(S_{lmp,q+s} + S_{lmp,q-s}\right)$$

$$S_{lmpq} \sin sM = \frac{1}{2}\left(S^{*}_{lmp,q+s} - S^{*}_{lmp,q-s}\right)$$

$$S^{*}_{lmpq} \cos sM = \frac{1}{2}\left(S^{*}_{lmp,q+s} + S^{*}_{lmp,q-s}\right) \qquad (69)$$

$$S^{*}_{lmpq} \sin sM = \frac{1}{2}\left(S_{lmp,q-s} - S_{lmp,q+s}\right)$$

Collecting the terms of order zero, we get :

$$\Delta r^{o}_{lmp} = A\left[\frac{2l-4p}{\dot{\psi}_{lmpo}} - \frac{g_{lp1}}{\dot{\psi}_{lmp1}} + \frac{g_{lp-1}}{\dot{\psi}_{lmp-1}}\right] S_{lmpo} \qquad (70)$$

(four terms, with S_{lmp2} and S_{lmp-2}, have cancelled out ...).

When $l-2p=o$ and $m=o$, the first term is actually zero (the factor $l-2p$ cancels the term before integration), and there remains a constant term :

$$\Delta r^{o}_{2p,o,p} = A\left[-\frac{g_{2p,p,1}}{\dot{\psi}_{2p,o,p,1}} + \frac{g_{2p,p,-1}}{\dot{\psi}_{2p,o,p,-1}}\right] \overline{C}_{2p,o}$$

This formula also shows that semi-major axis perturbations with frequencies $\dot{\psi}_{lmpo} = (l-2p)(\dot{\omega} + \dot{M}) + m(\dot{\Omega} - \dot{\theta})$ produce radial perturbations at the same frequencies ; it also implies that perturbations on a at any other frequency $(q \neq o)$ produce much smaller radial perturbations. It is also seen that the term in Δe and ΔM which yield the largest radial perturbations have the frequencies $\dot{\psi}_{lmp\pm 1}$ and produce terms at the same (previous) frequencies $\dot{\psi}_{lmpo}$. A major consequence of this is that the long period perturbations on *e* and *M* result in short period radial perturbations. For example, if $m=o$ and $l-2p=1$, we have perturbations on *e* and *M* with frequency $\dot{\omega}$, due to the odd degree zonals ; radially, they induce a perturbation with frequency $\dot{\omega} + \dot{M}$ (once per revolution). Due to the usually large amplitude of such long period perturbations on *e* and *M*, the short period radial perturbation on *r* is also quite large.

The terms of order 1 give a more complicated result. After some algebra, one finds :

$$\Delta r^1_{lmp} = Ae \Bigg[\left(\frac{C_1^{+1}}{\dot{\psi}_{lmp1}} + \frac{C_o^{+1}}{\dot{\psi}_{lmpo}} + \frac{C_2^{+1}}{\dot{\psi}_{lmp2}} + \frac{C_{-1}^{+1}}{\dot{\psi}_{lmp-1}} \right) S_{lmp1} + \left(\frac{C_{-1}^{-1}}{\dot{\psi}_{lmp-1}} + \frac{C_o^{-1}}{\dot{\psi}_{lmpo}} + \frac{C_{-2}^{-1}}{\dot{\psi}_{lmp-2}} + \frac{C_1^{-1}}{\dot{\psi}_{lmp1}} \right) S_{lmp-1} \left(\frac{C_1^{+3}}{\dot{\psi}_{lmp1}} + \frac{C_2^{+3}}{\dot{\psi}_{lmp2}} \right) S_{lmp3} + \left(\frac{C_{-1}^{-3}}{\dot{\psi}_{lmp-1}} + \frac{C_{-2}^{-3}}{\dot{\psi}_{lmp-2}} \right) S_{lmp-3} \Bigg] \quad (71)$$

Formula (70 is in agreement with previous works of other authors. In (71), we have :

$$C_1^{+1} = (3l - 4p + 1)(l - 2p + 3/2)$$

$$C_o^{+1} = \frac{3}{2}(p - l) - 1 + \frac{1}{4}(4p - 3l)(l - 4p) + \frac{3}{2}(l - 2p)n/\dot{\psi}_{lmpo}$$

$$C_2^{+1} = \frac{1}{2}\left[(p - l)\left(2l - 3p + \frac{5}{2}\right) - \frac{1}{4}(l - 2p + 2)^2 \right]$$

$$C_{-1}^{+1} = \frac{1}{4}(4p - l + 1)$$

$$C_{-1}^{-1} = (4p - l + 1)(l - 2p - 3/2)$$

$$C_o^{-1} = \frac{3}{2}p + 1 - \frac{1}{4}(4p - 3l)(l - 4p) - \frac{3}{2}(l - 2p)n/\dot{\psi}_{lmpo}$$

$$C_{-2}^{-1} = -\frac{1}{2}\left[p\left(3p - l + \frac{5}{2}\right) + \frac{1}{4}(l - 2p - 2)^2 \right]$$

$$C_1^{-1} = -\frac{1}{4}(3l - 4p + 1)$$

These are the terms found by Rosborough, with (it seems) a missing factor (1/2) for C_2^{+1} , C_{-1}^{+1}, C_1^{-1} and a factor (- 1/2) for C_{-2}^{-1}.

This author forgot the four other terms, which are :

$$C_1^{+3} = -\, g_{lp1}/2$$

$$C_{2}^{+3} = -\, g_{lp2}/2$$

$$C_{-1}^{-3} = g_{lp-1}/2$$

$$C_{-2}^{-3} = -\, g_{lp-2}/2$$

It is interesting to quote another form of formula (70), derived by Wagner (1985). Making the substitution of indices $(l,m,p) \to (m,k,l)$ already encountered in (66), denoting $\dot{\psi}_{lmpo} = \dot{\psi}_{kkm} = \dot{\psi}_{o}$ (then $\dot{\psi}_{lmp\pm 1} = \dot{\psi}_{o} \pm n$) and then $\dot{\psi}_{o}/n = \beta_{km} = k\left(1+\dot{\omega}/n\right) + m\left(\dot{\Omega}-\dot{\theta}\right)/n$, it is easy to transform (70) into :

$$\Delta r^{o} = a \sum_{m=o}^{L} \sum_{k=-L}^{+L} \sum_{\substack{l=\max(m,2,k) \\ l-k:even}}^{L} \left(\frac{R}{a}\right)^{l} \overline{F}_{lm,(l-k)/2}(I) \frac{\beta_{km}(l+1)-2k}{\beta_{km}\left(\beta_{km}^{2}-1\right)} S_{lm,(l-k)/2,o} \qquad (72)$$

3•2. GENERAL FORMULATION OF THE RADIAL PETURBATIONS

We start from (17) for r and we follow Rosborough (ibid) :

$$r = a \sum_{s=o}^{\infty} H_{s} \cos sM$$

with :

$$H_{o} = 1 + e^{2}/2$$

$$H_{s} = -\frac{2e}{s^{2}} \frac{d}{de}\left[J_{s}(se)\right], \quad \text{for } s > o$$

We define $H_{s}' = dH_{s}/de$, and find easily :

$$\Delta r = \Delta a \left(\sum_{s=o}^{\infty} H_{s} \cos sM\right) + a\,\Delta e \left(\sum_{s=o}^{\infty} H_{s}' \cos sM\right) - a\Delta M \left(\sum_{s=o}^{\infty} sH_{s} \sin sM\right) \qquad (73)$$

We then apply (64) and, for a particular set (l,m,p,q) we find :

$$\begin{aligned} \Delta r_{lmpq} &= \Delta a_{lmpq} \sum_{s} H_{s} \cos sM \\ &+ a\Delta e_{lmpq} \sum_{q} H_{s}' \cos SM \\ &- a\Delta M_{lmpq} \sum_{s} sH_{s} \sin sM \end{aligned} \qquad (74)$$

Looking at this series term by term, we have for any combination (l, m, p, q, s) :

$$\Delta r_{lmpqs} = \Delta a_{lmpq} H_s \cos sM + a\,\Delta e_{lmpq} H_s' \cos sM - a\Delta M_{lmpq} sH_s \sin sM \tag{75}$$

Since $J_s(x) = o(x^s), H_s$ is of order e^s, that is the perturbations decrease with s increasing. Replacing Δa_{lmpq} by $C^a_{lmpq} S_{lmpq}$, etc ... we get an expression with products $S_{lmpq} \cos sM$, $S_{lmpq} \sin sM$, $S^*_{lmpq} \sin sM$, which we transform by (69), hence :

$$\Delta r_{lmpqs} = \frac{1}{2}\left(C^a_{lmpq} H_s + aC^e_{lmpq} H_s' + aC^M_{lmpq} sH_s\right) S_{lmp(q+s)}$$
$$+ \frac{1}{2}\left(C^a_{lmpq} H_s + aC^e_{lmpq} H_s' - a\,C^M_{lmpq} sH_s\right) S_{lmpq(q-s)}$$

If the range of s is changed from o to $+\infty$ to be $-\infty$ to $+\infty$ and the functions $\widetilde{H}_s$ is defined as :

$$\widetilde{H}_o = H_o$$
$$\widetilde{H}_s = H_s/2\,, \widetilde{H}_{-s} = \widetilde{H}_s, (s = 1,2,\ldots+\infty),$$

then we can write in a compact form :

$$\Delta r_{lmpqs} = C_{lmpqs} S_{lmp(q+s)}$$

where :

$$C_{lmpqs} = C^a_{lmpq} \widetilde{H}_s + a\, C^e_{lmpq} \widetilde{H}_s' + a\, C^M_{lmpq}\, s\widetilde{H}_s$$

The total radial perturbation is written :

$$\Delta r = \sum_{l=2}^{\infty} \sum_{m=o}^{l} \sum_{p=o}^{l} \sum_{q=-\infty}^{+\infty} \sum_{s=-\infty}^{+\infty} \Delta r_{lmpqs}$$

We can make the following changes of indices :

$q' = q + s$: q' range is $-\infty, +\infty$

$s' = q$: s' range is $-\infty, +\infty$

and then rename q' as being q and s' as being s, and we obtain :

$$\Delta r = \sum_{l=2}^{\infty} \sum_{m=o}^{l} \sum_{p=o}^{l} \sum_{q=-\infty}^{+\infty} \Delta r_{lmpq} \tag{76}$$

where :

$$\Delta r_{lmpq} = C_{lmpq} S_{lmpq} \tag{77}$$

and :

$$\begin{aligned} C_{lmpq} &= \sum_{s=-\infty}^{+\infty} C_{lmps(q-s)} \\ &= \sum_{s=-\infty}^{+\infty} C^{a}_{lmps} \widetilde{H}_{q-s} + a\, C^{e}_{lmps} \widetilde{H}'_{q-s} + a(q-s) C^{M}_{lmps} \widetilde{H}_{q-s} \end{aligned} \tag{78}$$

This form shows that the radial perturbation amplitude at a given frequency $(l, m, p, q \text{ given})$ depends on the perturbations on *a, e, M* at that frequency ($s = q$ in the summation) and also of an infinite number of different frequencies (other values of *s*).

3•3. FREQUENCY SPECTRUM

We start from (77) and want to identify all terms of different frequency. From the form of S_{lmpq} and ψ_{lmpq}, we infer that we must distinguish between the zonal $(m = o)$ and non-zonal terms $(m > o)$. We found also simpler to start from a formula where the frequencies are indeed identified by three indices k, q, m (cf. formula (66) for $\mathcal{R}$) : $\psi_{kqm} = k(\omega + M) + qM + m(\Omega - \theta)$.

From this, it is clear that :

- when $m = o$: terms of all different frequencies are obtained for :
  . $k = o$:$+q$ and $-q$ but $q \neq o$ ($q = o$ gives a secular term)
  . $k \neq o$:(k, q) and $(-k, -q)$ since $\psi_{-k,-q,o} = -\psi_{kqo}$
- when $m > o$: all terms with different (k, q)'s generate different frequencies.

Hence, writing that we have a model truncated à $l = L$ and that *q* is limited to $|q| \leq Q$:

$$\Delta r = \sum_{m=o}^{L} \sum_{k=-L}^{+L} \sum_{q=-Q}^{+Q} \sum_{l=l_{\min}}^{L} C_{lm,(l-k)/2,q} S_{lm,(l-k)/2,q} \tag{79}$$

with $l_{\min} = \max(2, |k|, m)$ and $l - k$ being always even in the summation, we have :

- for the zonal terms :

. when $k = o$:

$$\sum_{q=1}^{Q}\left[\sum_{l=2}^{L}\left(C_{lo,l/2,q} + C_{lo,l/2,-q}\right)\overline{C}_{lo}\right]\cos qM$$

. when $k > o$

$$\sum_{q=-Q}^{+Q}\left[\sum_{l}\left(C_{lo,(l-k)/2,q} + (-1)^{l} C_{lo,(l+k)/2,-q}\right)\overline{C}_{lo}\right]\begin{pmatrix}\cos\\ \sin\end{pmatrix}_{k:odd}^{k:even}\psi_{kqo}$$

These two cases can be compacted in :

$$\begin{aligned}\Delta r_{(m=o)} &= \sum_{q=1}^{Q}\left[\sum_{j}\left(C_{2j,o,j,q} + C_{2j,o,j,-q}\right)\overline{C}_{2j,o}\right]\cos\psi_{oqo} \\ &+ \sum_{k=1}^{L}\sum_{q=-Q}^{Q}\left[\sum_{j}\left(C_{k+2j,o,j,q} + (-1)^{k} C_{k+2j,o,k+j,-q}\right)\overline{C}_{k+2j,o}\right]\begin{pmatrix}\cos\\ \sin\end{pmatrix}_{k:\text{odd}}^{k:\text{even}}\psi_{kqo}\end{aligned} \quad (80)$$

This has been derived by setting $l - k = 2j$, and we have j running from $j_{\min}$ to $j_{\max}$:

$$j_{\min} = \max\left(o, 1-\left[k/2\right]\right)$$

$$j_{\max} = [L-k]/2$$

- for the tesseral harmonics, using the same transformation of indices, we find :

$$\begin{aligned}\Delta r_{(m>o)} = \sum_{k=-L}^{+L}\sum_{q=-Q}^{+Q} &\left[\left(\sum_{j=j_{\min}}^{j_{\max}} C_{k+2j,m,j,q}\widetilde{C}_{k+2j,m}\right)\cos\psi_{kqm}\right. \\ &\left.+\left(\sum_{j=j_{\min}}^{j_{\max}} C_{k+2j,m,j,q}\,\widetilde{S}_{k+2j,m}\right)\sin\psi_{kqm}\right]\end{aligned} \quad (81)$$

where we now have :

$$j_{\min} = \max\left(o, 1-\left[k/2\right], -k, [m-k]/2\right)$$

$j_{\max} = [L-k]/2$, as before.

By letting the indices run as indicated, that is for $m = o$: $k = o$ to L, $q = -Q$ to Q $(q \neq o)$; and for $m > o$: $k = -L$ to $+L$, $q = -Q$ to $+Q$, we obtain all terms of different frequencies.

The amplitudes are obtained by :

$$.m = o, k = o: \left|\sum_{j}\left[\left(C_{2j,o,j,q} + C_{2j,o,j,-q}\right)\overline{C}_{2j,o}\right]\right|; q = 1 \text{ to } Q$$

$$.m = o, o < k \leq L: \left| \sum_j \left[C_{k+2j,o,j,q} + (-1)^k C_{k+2j,o,k+j,-q} \right] \overline{C}_{k+2j,o} \right| ; q = -Q \text{ to } +Q$$

$$.m > o, -L \leq k \leq L: \left[\left(\sum_j C_{k+2j,m,j,q} \, \widetilde{C}_{k+2j,m} \right)^2 + \left(\sum_j C_{k+2j,m,j,q} \, \widetilde{S}_{k+2j,m} \right)^2 \right]^{1/2} ; q = -Q \text{ to } +Q$$

As an example, figure 13 shows the spectrum of the Topex-Poseidon radial orbit perturbations based on Rapp 1991 model truncated at degree and order 70.

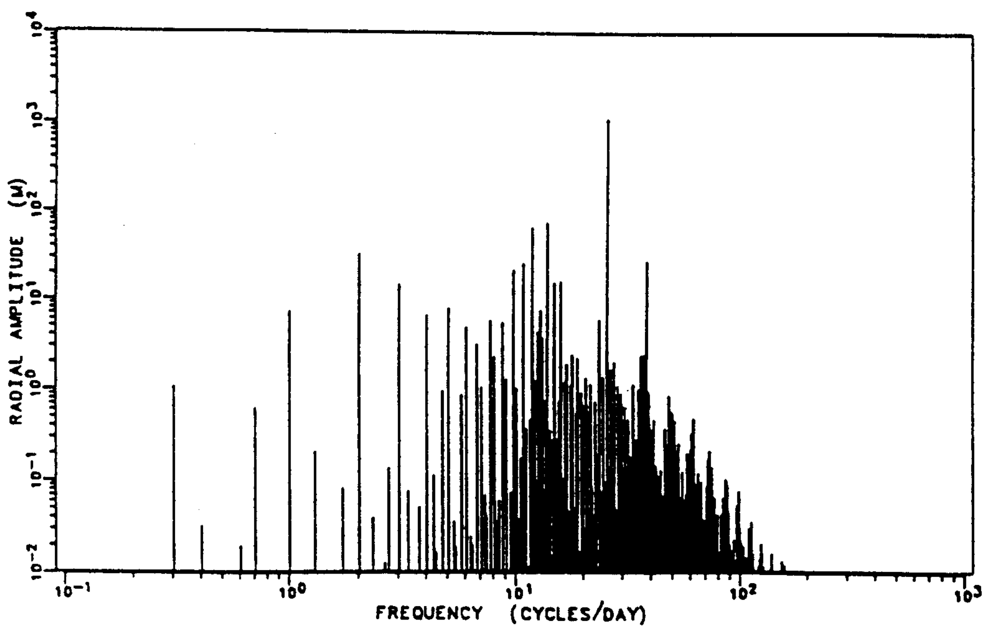

Fig. 13. Frequency spectrum of the Topex-Poseidon radial orbit perturbations

3•4. RADIAL PERTURBATIONS BY COEFFICIENT, BY ORDER, BY DEGREE

We will derive the r.m.s. perturbations for : each pair of coefficients $\left(\overline{C}_{lm}, \overline{S}_{lm}\right)$, then for all coefficients of a given order *m*, finally for all coefficients of a given degree. Depending on the case, we will use one form or another, such as (76), or (80)-(81), of the radial perturbations, which is most suited to identify the different frequencies in order to take properly the r.m.s. of the adh'oc terms.

3•4.1- For a pair of coefficients

We start from (76), for l and m fixed, that is :

$$\Delta r_{lm} = \sum_{p=o}^{l} \sum_{q=-Q}^{Q} C_{lmpq} S_{lmpq}$$

Then :

$$< \Delta r_{lm}^2 >= \sum_{p=o}^{l} \sum_{j=o}^{l} \sum_{q=-Q}^{Q} \sum_{s=-Q}^{Q} C_{lmpq} C_{lmjs} < S_{lmpq} S_{lmjs} >$$

Recalling the form of S_{lmpq}, it is clear that the means < ... > are zero unless $\psi_{lmpq} = \pm \psi_{lmjs}$. This condition is satisfied when :

$*m = o$: $j = p,\ s = q$ and $j = l - p,\ s = -q$; that is :

$$< S_{lopq}^2 > = \overline{C}_{lo}^2 < \begin{pmatrix} \cos^2 \\ \sin^2 \end{pmatrix} \psi_{lopq} > = \frac{1}{2} \overline{C}_{lo}^2$$

where < ... > is taken over the smallest common multiple of all encountered periods.

$< S_{lopq} S_{lo,l-p,-q} > = \overline{C}_{lo}^2 \begin{pmatrix} \cos \\ \sin \end{pmatrix} \psi_{lopq} \begin{pmatrix} \cos \\ \sin \end{pmatrix} \psi_{lo,l-p,-q}$ in distinguishing between l even (cos ...) and l odd (sin ...) ; hence $<...> = -(1)^l\, \overline{C}_{lo}^2 / 2$ in this case.

Therefore, for a zonal term :

$$< \Delta r_{lo}^2 > = \frac{1}{2} \overline{C}_{lo}^2 \sum_{p=o}^{l} \sum_{q=-Q}^{+Q} \left[C_{lopq}^2 + (-1)^l C_{lopq} C_{lo,l-p,-q} \right] \tag{82}$$

$*m > o$: we simply need $j = p,\ s = q$

Since $< S_{lmpq}^2 > = \frac{1}{2} \left(\overline{C}_{lm}^2 + \overline{S}_{lm}^2 \right)$ we have :

$$< \Delta r_{lm}^2 > = \frac{1}{2} \left(\overline{C}_{lm}^2 + \overline{S}_{lm}^2 \right) \sum_{p=o}^{l} \sum_{q=-Q}^{+Q} C_{lmpq}^2 \tag{83}$$

The r.m.s. follows by taking the square root of (82) or (83).

3•4.2- For a given order m

We already identified precisely the indices yielding different frequencies.

* $m = o$: we start from (80), square Δr_o and take the average ; hence :

$$< \Delta r^2_{(m=o)} > = \frac{1}{2}\sum_{q=1}^{Q}\left[\sum_j \left(C_{2j,o,j,q} + C_{2j,o,j,-q}\right)\overline{C}_{2j,o}\right]^2 + \frac{1}{2}\sum_{k=1}^{L}\sum_{q=-Q}^{Q}\left[\sum_j \left(C_{k+2j,o,j,q} + (-1)^k C_{k+2j,o,k+j,-q}\right)\overline{C}_{k+2j,o}\right]^2 \quad (84)$$

* $m > o$: starting from (81), we readily find :

$$< \Delta r^2_{(m>o)} = \frac{1}{2}\sum_{k=-L}^{L}\sum_{q=-Q}^{Q}\left[\left(\sum_j C_{k+2j,m,j,q}\widetilde{C}_{k+2j,m}\right)^2 + \left(\sum_j C_{k+2j,m,j,q}\widetilde{S}_{k+2j,m}\right)^2\right] \quad (85)$$

In (84) and (85), the range of index j is as prescribed in (80) and (81). From these, it is easy to find the full field perturbations in summing over all orders (since frequencies of all terms of different orders are all different), that is :

$$r.m.s.(\Delta r) = \left[\sum_{m=o}^{L} < \Delta r^2_{(m)} >\right]^{1/2} \quad (86)$$

3•4.3- For a given degree l

We here start from (76), for l fixed, that is :

$$\Delta r_{(l)} = \sum_{m=o}^{l}\sum_{p=o}^{l}\sum_{q=-Q}^{+Q} C_{lmpq} S_{lmpq}$$

Then :

$$< \Delta r^2_{(l)} > = \sum_{m=o}^{l}\sum_{k=o}^{l}\sum_{p=o}^{l}\sum_{j=o}^{l}\sum_{q=-Q}^{Q}\sum_{s=-Q}^{Q} C_{lmpq} C_{lkjs} < S_{lmpq} S_{lkjs} >$$

If $m \neq k$, frequencies are necessarily different and < ... > is zero. Hence we are left with :

$$< \Delta r_{(l)}^2 > = \sum_{m=o}^{l} \sum_{p=o}^{l} \sum_{j=o}^{l} \sum_{q=-Q}^{Q} \sum_{s=-Q}^{Q} C_{lmpq} C_{lmjs} < S_{lmpq} S_{lmjs} >,$$

which is nothing but :

$$< \Delta r_{(l)}^2 > = \sum_{m=o}^{l} < \Delta r_{lm}^2 > \qquad (87)$$

Therefore it suffices to add the terms of (82) and (83) for l fixed.

As examples, fig. 14 and 15 give the radial perturbations by order and by degree which are computed for Topex (model truncated at degree and order 50).

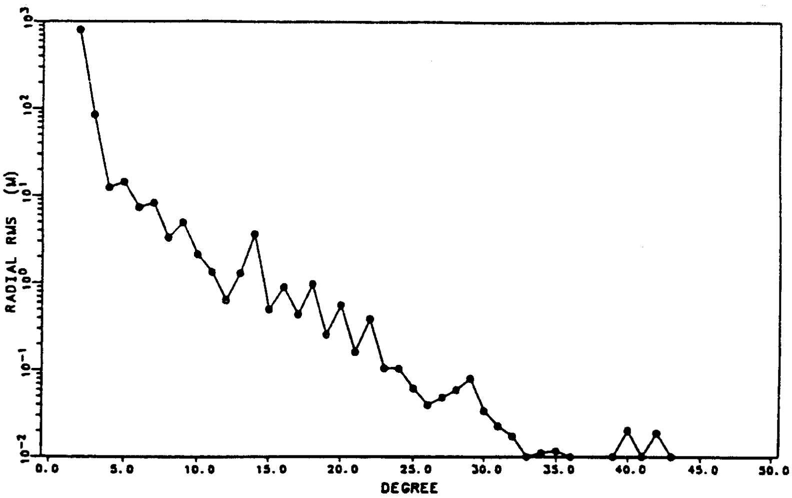

Fig. 14. r.m.s. of the Topex-Poseidon radial orbit perturbations by degree

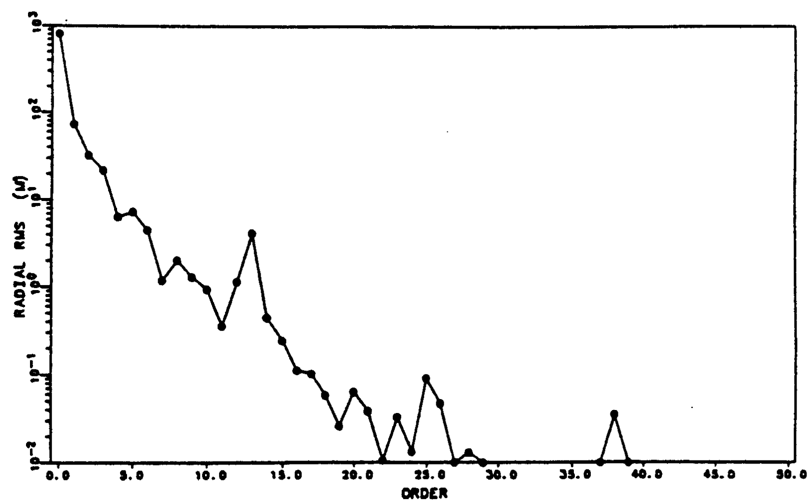

Fig. 15. r.m.s. of the Topex-Poseidon radial orbit perturbations by order

4• CONCLUSION

We have given the readers the basic tools to evaluate the static gravitational perturbations of the Earth on a mean satellite orbit, when a gravitational model is given as a series of spherical harmonic coefficients. These perturbations have been explicitely written for the elliptic mean orbital elements, as well as the magnitude of the radius vector which variations and spectral components are fundamental quantities involved in satellite altimetry.

For further investigation of gravity field recovery capabilities of some satellite systems, it is also necessary to have the perturbations on the transverse and normal components of the radius vector, and sometimes on other quantities, too (e.g. gravity gradients). Adopting the notations $(\Delta r, \Delta\tau, \Delta\zeta) = (\Delta u, \Delta v, \Delta w)$ already used in the derivation of the Hill equations, and assuming a quasi-circular mean orbit, we have the following non resonant perturbations (cf. Balmino and Perosanz, 1995) :

$$\begin{bmatrix} \Delta u \\ \Delta v \\ \Delta w \end{bmatrix} = a \sum_{m,l,k[2]} \left(\frac{R}{a}\right)^l \frac{1}{\beta_{km}^2 - 1} \begin{bmatrix} Q_u\ \tau_{lmk} \\ Q_v\ \tau^*_{lmk} \\ Q_w\ \tau^*_{lmk} \end{bmatrix} \tag{88}$$

where :

$$\begin{aligned} Q_u &= \overline{F}_{l,m,(l-k)/2}\ \left[\beta_{km}(l+1) - 2k\right] \big/ \beta_{km} \\ Q_V &= \overline{F}_{l,m,(l-k)/2} \left[2\beta_{km}(l+1) - k\left(3+\beta_{km}^2\right)\right] \big/ \beta_{km}^2 \\ Q_w &= \frac{1}{2}\left(\overline{D}^*_{l,m,k-1} - \overline{D}^*_{l,m,k+1} - \overline{E}^*_{l,m,k-1} - \overline{E}^*_{l,m,k+1}\right) \end{aligned} \tag{89}$$

and

$$\begin{aligned} \tau_{lmk} &= \widetilde{C}_{lm} \cos X_{km} + \widetilde{S}_{lm} \sin X_{km} \\ \tau^*_{lmk} &= \widetilde{S}_{lm} \cos X_{km} - \widetilde{C}_{lm} \sin X_{km} \\ \overline{D}^*_{lmk} &= d\left[\overline{F}_{l,m,(l-k)/2}\right] \big/ dI \\ \overline{E}^*_{lmk} &= \overline{F}_{l,m,(l-k)/2}\,(k\cos I - m) / \sin I \end{aligned} \tag{90}$$

In those expressions, the quantities : $R, a, I, \overline{F}_{lmp}, \widetilde{C}_{lm}$ and $\widetilde{S}_{lm}$ are as previously defined. Furthermore we have :

$$X_{km} = k(\omega + M) + m(\Omega - \theta)$$
$$\beta_{km} = \dot{X}_{km} / n \qquad (91)$$

As before, n is the orbital mean motion. β_{km}, already introduced in formula (72) which gave Δu, is therefore a frequency expressed in cycle per revolution ; its module is different from 1 in our case (non resonant). The summations in equation (88) run from $m = o$ to L, $l = \max(m,2)$ to L, and $k = -l$ to $+l$, where L is the maximum degree and order at which the gravitational potential series are truncated. In addition, one must have $l - k$ even for $\Delta u, \Delta v$ and $l - k$ odd for the Δw component. Of course the perturbations in velocity may be obtained by simply derivating those expressions with respect to time.

Other functions of the perturbations may be of interest, for instance the variation ΔV of the total velocity. Using the kinetic energy equation (6) we find :

$$\Delta V = \mu / V \left(1/2 \; \Delta a / a^2 - \Delta r / r^2 \right) \qquad (92)$$

Δa is taken from equation (64) and $\Delta r = \Delta u$ from equation (88) above. The result is, again for a quasi-circular orbit :

$$\Delta V = \sum_{m,l,k[l-k:\text{even}]} a \left(\frac{R}{a} \right)^l \frac{1}{\beta_{km}^2 - 1} Q_V \; \tau_{lmk} \qquad (93)$$

with :

$$Q_V = n \overline{F}_{l,m,(l-k)/2} \left[k \left(1 + \beta_{km}^2 \right) - (l+1) \beta_{km} \right] / \beta_{km} \qquad (94)$$

Also of interest is the relative velocity perturbation between two co-orbiting spacecraft separated by a mean angular distance α. This perturbation, $\Delta \dot{d}$, is expressed when α is small $(\leq 10°)$ by :

$$\Delta \dot{d} = \sum_{m,l,k[l-k:\text{even}]} a \left(\frac{R}{a} \right)^l \frac{1}{\beta_{km}^2 - 1} Q_{\dot{d}} \; \tau'_{lmk} \qquad (95)$$

where :

$$Q_{\dot{d}} = 2 Q_V \sin k\alpha / 2$$
$$\tau'_{lmk} = \widetilde{S}_{lm} \cos X'_{km} - \widetilde{C}_{lm} \sin X'_{km} \qquad (96)$$

and

$$X'_{km} = X_{km} + k\alpha/2 \tag{97}$$

Finally, gravity gradient observations on board a spacecraft may be carried out, solely or in combination with other measurements. For completeness, we just recall below the expressions of the diagonal terms of the gravity gradient tensor in the local orbital frame, still in the case of a quasi-circular orbit :

$$\begin{bmatrix} \Gamma_{uu} \\ \Gamma_{vv} \\ \Gamma_{ww} \end{bmatrix} = n^2 \sum_{m,l,k[2]} \left(\frac{R}{a}\right)^l \overline{F}_{lm,(l-k)/2} \begin{bmatrix} (l+1)(l+2) \\ -(l+1+k^2) \\ k^2-(l+1)^2 \end{bmatrix} \tau_{lmk} \tag{98}$$

Equations (88) and their time derivatives, and equations (93), (95) and (98) allow to perform sensitivity analysis of all the observation systems considered in modern satellite projects aimed at mapping the Earth's gravity field, such as those discussed in tutorial # 5.

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