A simple anisotropic model of the covariance function of the terrestrial gravity field over coastal areas

Jonathan Chenal* and Jean-Pierre Barriot*

* Service de la Géodésie et du Nivellement, Institut Géographique National, Saint-Mandé, France.
** Bureau Gravimétrique International, Toulouse, France.

I. Introduction

The modelling of the gravity and of the geoid at the interface between continents and oceans is a very difficult challenge, as the gravity field over these regions exhibits large anisotropies. We consider in this short paper how to build simple 2D anisotropic covariance functions for collocation purposes. A detailed report is available upon request at BGI (Chenal, 2004).

II. A simple 2D geophysical model of the littoral

We consider a very simple 2D model, where the continental crust (Boillot, 1982) can be considered as a box invariant by translation along the littoral line (the $\vec{j}$ axis perpendicular to the sheet, see Fig. 1)

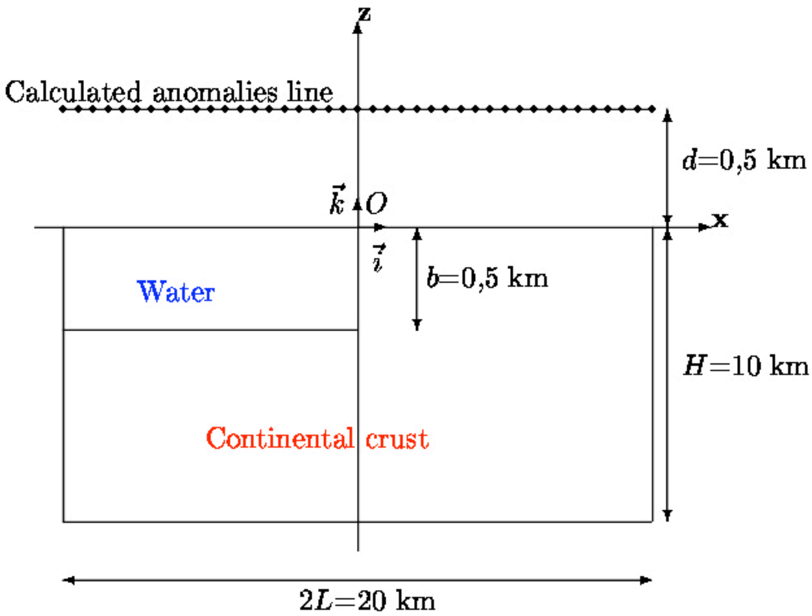

Fig. 1: 2D-model of the littoral: the sources of the gravity field are located in the box $-H < z < 0$ and $-L < x < L$ (the water slice being excluded). The gravity anomalies are computed on the dotted line, from left to right.

We also consider 2D sources (i.e. sources with infinite length along $\vec{j}$). This leads to

$$\Delta g_{Tot;z}(x', d) = G \sum_{i=1}^{N} \frac{\Delta m_i \, (d - z_i)}{(d - z_i)^2 + (x' - x_i)^2} \qquad [1]$$

where $\Delta g_{Tot;z}$ is the gravity anomaly along the vertical z, (x_i, y_i) is the location of each line mass, and x' is the abscissa coordinate along $\vec{i}$.

We suppose that the sources can be either randomly uniformly distributed or following a mesh of cell size e. In both cases, we use the same absolute numerical value for the masses, but with a sign uniformly distributed, to reduce computation costs.

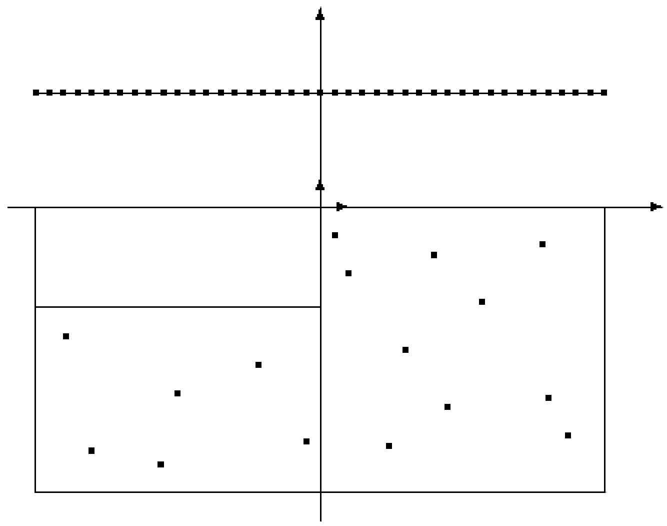

Fig. 2: Random distribution of the sources

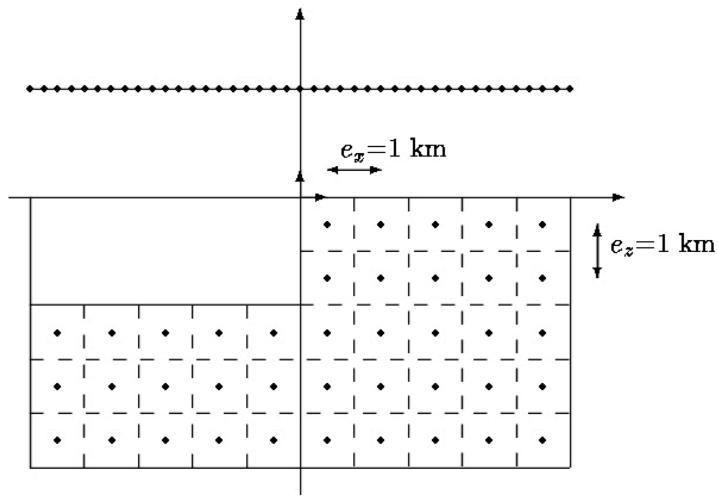

Fig. 3: Network distribution for the sources

If we consider N realizations (drawings) of the sources, the expectation of the related gravity anomaly is (Pelat, 2003):

$$E\{\Delta g_{Tot}(x', d)\} = \frac{1}{N_T} \sum_{j=1}^{N_T} \Delta g_{Tot}^{(j)}(x', d)$$

[2]

Besides, if we note

$$\Gamma^{(j)} = \begin{pmatrix} \Delta g^{(j)}_{Tot;z}(x'_1, d) \\ \vdots \\ \Delta g^{(j)}_{Tot;z}(x'_p, d) \\ \vdots \\ \Delta g^{(j)}_{Tot;z}(x'_r, d) \end{pmatrix}$$

$$\overline{\Gamma} = \begin{pmatrix} E\{\Delta g_{Tot;z}(x'_1, d)\} \\ \vdots \\ E\{\Delta g_{Tot;z}(x'_p, d)\} \\ \vdots \\ E\{\Delta g_{Tot;z}(x'_r, d)\} \end{pmatrix}$$

[3] and [4]

then the covariance matrix of the gravity anomaly over the N realizations is

$$V = E\{\Gamma\, \Gamma^T\} - \overline{\Gamma}\, \overline{\Gamma}^T$$

[5]

The most practical way to verify that this matrix is positive definite (as it should be for a covariance matrix) is to try to decompose it numerically by the Cholesky method (this also apply to the model matrices, see thereafter).

We have chosen by trials and errors two mathematical models for the anisotropic covariance function between two any points P and Q, the first one being

$$C_1(P,Q) = C_0\, exp\left(-\frac{(x_P - x_Q)^2}{A_e^2}\right)\ \left[1 + \frac{|b|}{P_h}\ \tanh\left(\frac{x_P + x_Q}{A_t}\right)\right]$$

[6]

with the free parameters C_0, A_e, P_h and A_t. The parameter x is the abscissa coordinate (see Fig. 1). The hyperbolic tangent is introduced to fit the behaviour of the covariance along the diagonal of the matrix (see thereafter), and the exponential introduces the usual decay with the distance. The second model is

$$C_2(P,Q) = C_0\, exp\left(-\frac{|x_P - x_Q|}{A_e}\right)\ \left[1 + \frac{|b|}{P_h}\ \tanh\left(\frac{x_P + x_Q}{A_t}\right)\right]$$

[7]

We note that the behaviour in the region over the sea must be equal to the one in the terrestrial region when the depth of the water is null. (i.e. when b =0).

We give the first model as information, as we rapidly discovered that it is was leading to inconsistent models of covariance matrices (i.e. non positive definite), and so we focused on the second model.

To compare the matrix models M_1 to the pseudo-experimental covariance matrices M_2, we have chosen, also after trials and errors, to minimize the criterion

$$R_M(P,Q) = \frac{|M_1(P,Q) - M_2(P,Q)|}{M_1(P,Q) + M_2(P,Q)}$$

[8]

with respect to C_0; P_h, A_t and A_e. As this criterion is strongly non-linear, the minimization was done by scanning the whole set of free parameters, which is permitted by their small number N_T.

III. Results

We first verified that the expectations of the gravity anomalies were zero over the computation profile (see Fig. 1) for a sufficient number of drawings (see Fig. 4 below).

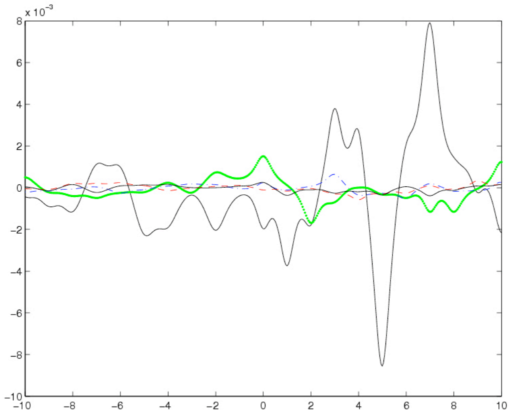

Fig. 4: Numerical values of the gravity anomalies expectations for different values of N_T (black line: 100; dotted green: 1 000; semi-dotted, semi-dashed blue: 5 000; the dotted red: 10 000; small black: 20 000. The x-coordinates are in km, the y ones in 10^{-1} mgal. A random distribution of sources was used. A similar result holds for a network distribution.

We can then consider that the gravity anomalies expectations are centred for N_T = 20 000, and we retained this value for the following computations.

III-a Results for a network distribution of sources

The "pseudo-experimental" covariance matrix obtained for the network distribution is then as follows, from formulas [3] and [4]:

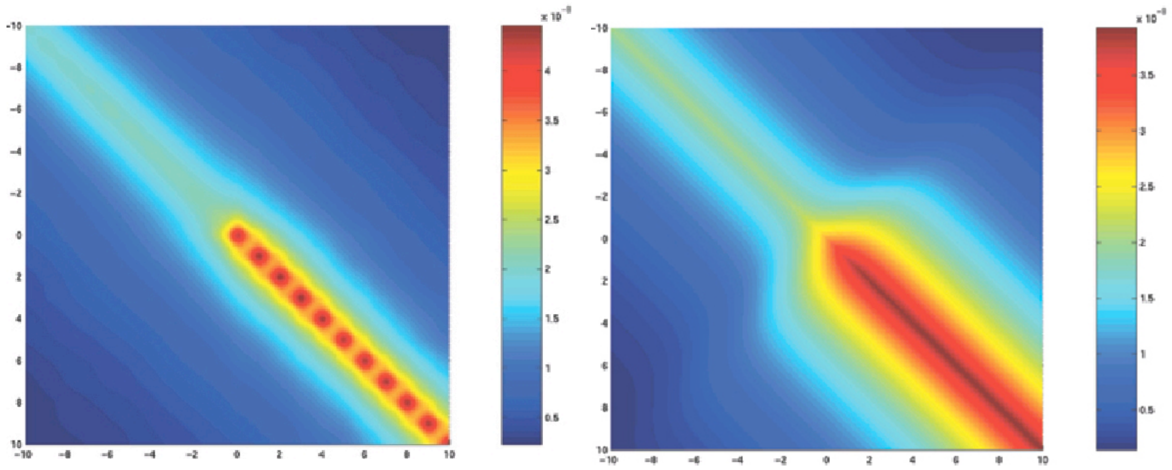

Fig. 5: "pseudo-Experimental" (left) and fitted (right) covariance matrices coming from a network of linear sources. The fit is obtained with $C_0=3.0\ 10^{-8}$; $A_t=1.93$ km; $P_h=1.55$ km; $A_e=6.92$ km.

One can note in Fig. 5-Left that, as expected, the variances in the region over the sea (upper left corner) are lower than in the terrestrial region (lower right corner). The local maxima over the terrestrial zone correspond to mass anomalies situated just above the gravity anomalies evaluation points. The best fit for this "pseudo-experimental" matrix, by using formula [7], is shown in Fig. 5-Right.

For easy comparisons, we also defined some cuts in these matrices, as:

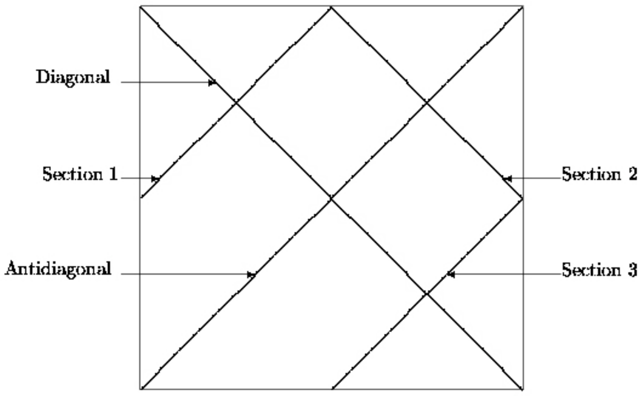

Fig. 6: Definition of the cuts over the covariance matrices.

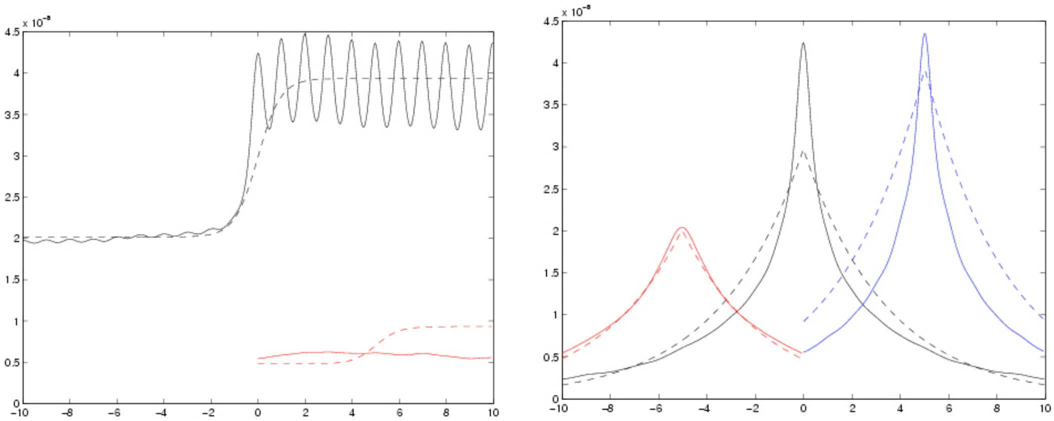

Fig. 7: Comparison over the cuts defined on Fig. 6 of the "pseudo-experimental" (plain lines) and model covariance matrices (dashed lines) for a network distribution of sources. Left: diagonal (black lines), and section 2 (red lines). Right: antidiagonal (black lines), section 1 (red lines) and section 3 (blue lines).

We can see on Fig. 8 that the fit (hyperbolic tangent) along the diagonal is very acceptable, except for the undulations caused by shallow network sources. The antidiagonal behaviour is also quite good.

III-b Results for a random distribution of sources

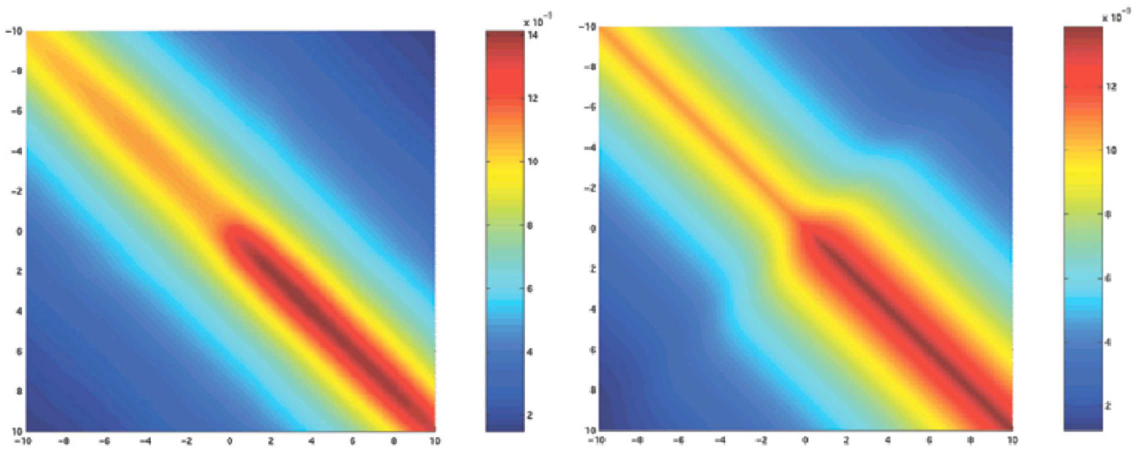

Fig. 8: "Pseudo-experimental" (left) and fitted (right) covariance matrices coming from a random distribution of linear sources. The fit is obtained with C_0=1.23 10^{-8}; A_t=1.43 km; P_h=3.88 km; A_e=8.70 km.